A VIBRATION ANALYSIS SOFTWARE FOR LAMINATED COMPOSITE AXISYMMETRIC SHELLS AND BEAMS

by

VIRAJ L. HATGADKAR

DEPARTMENT OF MECHANICAL ENGINEERING

INDIAN INSTITUTE OF TECHNOLOGY, KANPUR

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VIRAJ L. HATGADKAR

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CERTIFICATE

This is to certify that the present work "A vibration analysis software for laminated composite axisymmetric shells and beams" has been carried out by Mr. Viraj L. Hatgadkar under my supervision and it was not been submitted elsewhere.

B. Sahay
(B. Sahay)
Professor
Department of Mechanical Engineering
I.I.T. KANPUR - 208 016

Dedicated
to my
parents

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(Viraj L. Hatgadkar)

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NOMENCLATURE

BEAM

L	length of beam
B	width of beam
h	height (thickness) of beam
X,Y,Z	global co-ordinates
u,v,w	displacement along the X,Y,Z axes
u_o	displacement
$\varepsilon_x, \varepsilon_y, \varepsilon_z$	strains
$\sigma_x, \sigma_y, \sigma_{xy}$	stresses
$\bar{Q}_{11}, \bar{Q}_{12}, \bar{Q}_{16}$	stiffness coefficients (Appendix - 1)
N_x, M_x	force and moment aggregate
ρ	density of composite
x	local co-ordinate
e	length of element
$N_1 \ldots N_4$	shape functions
α	natural frequency
Q_1, Q_2	shear forces
M_1, M_2	moments
$\bar{\omega}$	forcing frequency
$\bar{F}$	nodal force vector
Ω	non-dimensional frequency parameter

AXISYMMETRIC SHELLS

Θ, Y, R	global co-ordinates
ξ, η, ζ	local co-ordinates of the element
N_1, N_2, N_3	shape functions
t_i	thickness at node i
$\bar{V}_1, \bar{V}_2, \bar{V}3$	unit vectors along ξ, η, ζ directions
i, j, k	unit vectors along Θ, Y, R direction
E_L, E_T	modulus of elasticity along L and T directions
G_{LT}	modulus of rigidity in L-T plane
h_k	thickness of K^{th} layer
i	subscript used for nodes 1, 2 and 3
L	direction parallel to fibre
n	total number of layers
S_{ij}	elements of stiffness matrix
t	thickness of shell
T	direction normal to fibre
u, v, w	displacements at any point in the shell along Θ, Y, R directions
$[\ ]^T$	transpose of the matrix
$[D]$	stress-strain matrix
$[B]$	strain-displacement matrix
$\lvert J \rvert$	determinant of Jacobian matrix
$[S]$	strain-stress matrix
$[T]$	co-ordinate transfer matrix
α	rotation of the normal to the shell in Y-R plane
β	angle between η and L
γ	displacement vector of the element
ε	normal strain

σ	normal stress
τ	shear stress
γ	shear strain
ν_{LT}	Poisson's ratio measuring normal strain in T direction under uniaxial normal stress along L direction
$[K]$	elemental stiffness matrix
$[M]$	elemental mass matrix
ρ_C	density of the composite
ω	natural frequency
Ω	non-dimensional frequency

EPITOME

A Computer Aided Design (CAD) software has been developed for the vibration analysis of Fibre Reinforced Plastic (FRP) laminated composite beams and axisymmetric shells of revolution.

For beams, laminate composite theory is used. The finite element formulation for the vibration analysis of the laminated beams is presented.

The formulation includes arbitrary number of bonded layers each of which may have different thicknesses and orientation of the elastic axes.

In the case of laminated axisymmetric shells, isoparametric elements with four degrees of freedom per node including the normal rotation are used. The use of isoparametric element considers anisotropy across the thickness. The deformation of the normal to the mid-surface is included. Strain along the thickness is considered. The formulation includes arbitrary number of bonded layers each of which may have different thicknesses, orientation of elastic axes. The formulation is compared for cylindrical shells with Flugge's theory and refined theory [2]. Using this formulation, axisymmetric shells like conical shells, shells with Gaussian curvature and spherical caps can be analysed for axisymmetric modes of vibration.

In the present software, analysis of free vibrations for natural frequencies and mode shapes and analysis of the response of sinusoidal forced vibrations are included.

CHAPTER I

INTRODUCTION

Composite materials is the latest field in the development of the special purpose materials. These materials are replacing the conventional materials in every possible way. This is due to the fact that composite materials can be fabricated according to the requirements of the design to meet the mechanical, electrical, thermal and other parameters. Present work deals with the field of fibre composites. The success of fibre composites results from the ability to make use of the outstanding strength, stiffness and low specific gravity of fibres such as glass, graphite, Kevlar etc. These outstanding mechanical properties are combined with the unique flexibility in the design capability and ease of fabrication.

In the present work, a computer Aided Design (CAD) software is developed for vibration analysis of Fibre Reinforced Plastic (FRP) laminated composite beams and axisymmetric shells of revolution. Finite Element Method (FEM) is used. Analysis of free vibrations for natural frequencies and mode shapes and analysis of the response of sinusoidal forced vibrations are included in the software.

LITERATURE SURVEY

A detailed literature survey has been done on the following subjects.

Free Vibrations of Laminated Composite Axisymmetric Shells

The composite laminated shells have found extensive applications in the aircraft and space craft industries. Considerable progress has been made during the last decade in applying the FEM for the analysis of shell structures. The FEM applications in this area may be classified into two types:

1. Those in which triangular or quadrilateral elements have been used [10].

2. Those in which ring elements have been used [3,8,11].

The first approach permits general treatment of the shell and necessitates the use of large number of degrees of freedom to simulate adequate inter-element compatibility.

In the second approach certain known displacement distributions along the circumferential direction are chosen and as a result, the inter-element compatibility is easily achieved. But this introduces a limitation in the use of this element, for the basic shell problem, as it has to be reduced to a set of problems with suitable circumferential distributions. Generally a Fourier distribution of loads and displacements is considered.

These two types of elements are good for the analysis of frequencies in circumferential modes. For axisymmetric mode frequencies, due to large number of elements, the problem becomes complicated. Till now, most of the work on the vibrations of

laminated composite shells is on circumferential mode frequency analysis. Application of FEM to axisymmetric mode frequency analysis is few.

Analysis of circumferential mode frequencies is done in references [4] to [17]. In ref. [4], [5] free vibration problem for circular and non-circular cross-ply laminated composite shells is studied. Shivakumar et.al [8] have used eight degrees of freedom per node for a typical conical laminated shell ring element. Wu et.al [9] used higher order curved shell elements. Bert et.al [10] studied the effect of various shell theories, thickness shear flexibility and bimodulus action on small amplitude vibration of thick circular shells laminate of bimodulus composite materials. Sheinman et. al [11] investigated that coupling between symmetric and antisymmetric modes is a function of stacking combination & orientation.It's influence decreases as the number of layers in the laminate increases. The influence of the coupling effect on natural frequencies decreases as the lowest frequency is approached. Sheinman et.al [12] applied a linear theory. Stavsky [13] used refined Love-type theory. Bert et.al [14] examined the non-symmetric modes of an arbitrarily layered general anisotropic finite length shell. Dong [15] used Donnel theory for free vibrations of laminated orthotropic cylindrical shells. Greenberg et.al [16] used Love-type theory for layered shells in which each layer can have fixed fibre orientation. Bhimaraddi [17] concluded that the stress and displacement distribution across the

thickness of laminated composite shells is non-linear. This non-linearity increases with increase in the thickness and the material orthotropy.

The analysis of axisymmetric mode frequencies is done in reference [2],[3],[18].

In ref. [2] Sun et.al developed a refined theory for vibration of laminated composite thick shells. This theory includes the effects of the transverse shear deformation, transverse normal stress and strain, rotary inertia and higher order stiffness terms. Sankaranarayanan et.al [3] studied axisymmetric free vibrations of laminated conical shells with linear thickness variation in the meridional direction. Rayleigh-Ritz procedure is adopted for the analysis. Classical thin shell theory is used. Parametric studies are presented to illustrate the effects of geometric, material and coupling parameters and of the boundary conditions on the frequencies and mode shapes. Malhotra et.al [18] used a superparametric curved element with four degrees of freedom per node including normal rotation for the vibration analysis of filament wound FRP shells of revolution. Effect of winding angle, metallic liner and material on frequencies is studied.

Forced Vibrations of Laminated Composite Axisymmetric Shells

Wu et.al [9] used higher order curved shell elements to investigate forced responses of thin laminated composite shells. The effect of geometric non-linearity was included in the determination of forced response using Newmark's integration

scheme. Reddy [28] presented an extension of the Sander's shell theory for doubly curved shells to shear deformation theory of laminated shells. The theory accounts for transverse shear strains and rotation about the normal to the shell mid-surface. Exact solutions for simply supported, doubly curved, cross-ply laminated shells under sinusoidal, uniformly distributed and concentrated point load at centre are presented.

Free Vibrations of Laminated FRP Composite Beam

Miller et.al [23] presented an analytical method for determination of resonant frequencies of generally orthotropic beams where flexural to torsional coupling is caused by the character of the elastic material constants. The analysis predicts that torsional as well as flexural resonant conditions may exist for a freely vibrating beam. Teoh et.al [24] used a continuous model which includes shear and rotary interia. Abarcar et.al [25] have formulated a discrete mathematical model based on a combination of Myklestad and Holzer's method of lumped mass to predict the natural frequencies and mode shapes of Fixed-free beams of general orthotropy. The model incorporates both transverse shear deformation and rotary interia effects. Teh et.al [22] presented two finite element models for the prediction of natural frequencies of fixed-free beams of general orthotropy. The discrete models include the transverse shear deformation and the rotary interia effects. Teh et.al [27] studied the effect of fibre orientation on free vibrations of composite beams. The torsion-flexure coupling effect of generally orthotropic beams is dependent on reinforcing fibre orientation and mode order. At higher ranks

of vibration this coupling effect is principally contributed by the twisting moment induced due to bending. The influence of fibre orientation on normal mode shape is more significant for small values of fibre orientation.

The work cited above is on generally orthotropic beams. The survey indicates that not much literature is available on the vibration of laminated composite beams. Singh [26] has used finite difference method for laminated composite cantilevers for finding natural frequencies of beams with flaws and without flaws. He has also experimentally verified the results.

PRESENT WORK

The following problems have been attempted in the present work:

1. Free vibrations of laminated composite beams.
2. Sinusoidal forced vibrations of laminated composite beams.
3. Free vibrations of laminated composite axisymmetric shells.
4. Sinusoidal forced vibrations of laminated composite axisymmetric shells.

Laminated FRP composite beams with following boundary conditions are analysed.

(a) Simply supported

(b) Fixed – Free

(c) Fixed – Fixed

Laminated composite theory is applied for formulating the finite element model. Galerkin method is used. The formulation

considers laminated beam with arbitrary number of layers and different thicknesses having arbitrary orientation of elastic axes. If the laminate beam is constructed of stacking a number of orthotropic laminae in an arbitrary sequence of orientations the undesirable effects of coupling between bending and stretching will be present. To avoid this , for each ply above the midplane an identical ply should be placed at equal distance below the midplane. This type of laminate is called symmetric laminate. Use of symmetric laminate will eliminate warping due to inplane loads.

According to the Love-Kirchhoff hypothesis,the normal to the middle plane remains normal to the deformed surface and moreover there is no stretching along these normals. This Love-Kirchhoff theory is applied to the present formulation. It is also considered that the plane sections remain plane during the deformation. Thus bending-torsion coupling is not considered in the present formulation.

In the formulation of laminated composite axisymmetric shells the effect of anisotropy along the thickness is considered. This anisotropy arises due to stacking of orthotropic laminae with different orientations in a laminate. Therefore Love-Kirchhoff theory is not applicable specially in the case of thick shells.

An attempt is made in present work to refine the mathematical model by discarding Love-Kirchhoff theory and considering the following assumptions.

(i) Normal to the middle surface does not remain straight after deformation.

(ii) Distribution of displacements is non-linear and that of strain is linear, across the thickness.

(iii) Normal to the middle surface does not remain normal after deformation.

(iv) Case I: Strains in the direction normal to the mid-surface are considered.

Case II: Strains in the direction normal to the mid-surface are not considered.

In the finite element analysis shear deformation and coupling effects are included. An isoparametric element is used to include the effect of non-linearity of displacement across the thickness. Using this formulation a shell with arbitrary number of bonded layers having different thicknesses and orientation of elastic axes can be analysed. The element has four degrees of freedom per node including normal rotation. As the formulation is for a general second order parabolic curve, various axisymmetric shells like cylinder, cone, hyperboloid, cylinders with Gaussian curvature and spherical caps etc. can be analysed for axisymmetric mode vibration.

Two types of boundary conditions are considered.

(a) Simply supported without axial constraint.

(b) Simply supported with axial constraint.

CHAPTER II

ANALYSIS OF LAMINATED COMPOSITE BEAMS AND AXISYMMETRIC SHELLS

LAMINATED BEAM

As shown in Fig. 2.1, the X, Y, Z axes are along length, width and thickness of the beam. Let length, breadth and height of the beam be L, B and h respectively. The deformation be u,v,w along X,Y,Z axes respectively.

The bending deformation is in X-Z plane as shown in Fig. 2.2. Hence, v = 0 and u and w are constant across width. Stretching of the normal ABCD (Fig. 2.2) is negligible. Inplane strains are,

$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{\partial u_o}{\partial x} - Z\,\frac{\partial^2 w}{\partial x^2} \qquad \text{(a)}$$

$$\varepsilon_y = 0, \quad \gamma_{xy} = 0 \qquad \text{(b)} \qquad (2.1)$$

where,

$$u = u_o - \frac{\partial w}{\partial x}\, Z \qquad \text{(c)}$$

$u_o \rightarrow$ displacement of the middle plane.

In plane stresses are,

$$\begin{aligned} \sigma_x &= \bar{Q}_{11}\ \varepsilon_x \\ \sigma_y &= \bar{Q}_{12}\ \varepsilon_x \\ \sigma_{xy} &= \bar{Q}_{16}\ \varepsilon_x \end{aligned} \qquad (2.2)$$

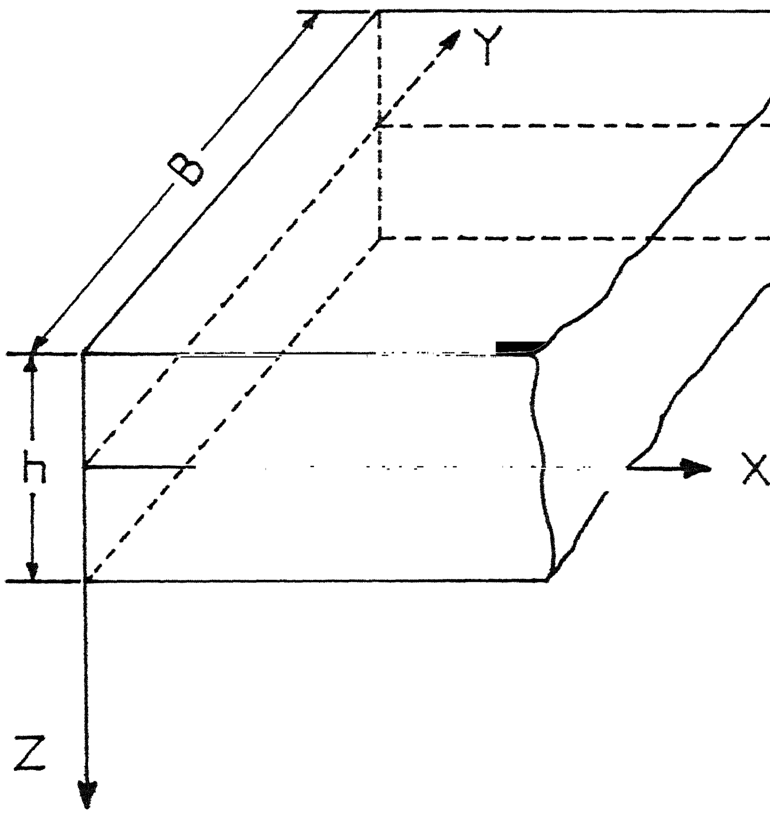


Fig. 2·1

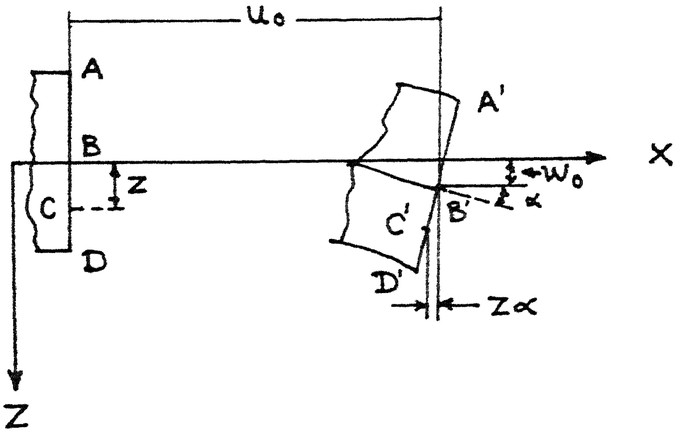


Fig. 2·2 .

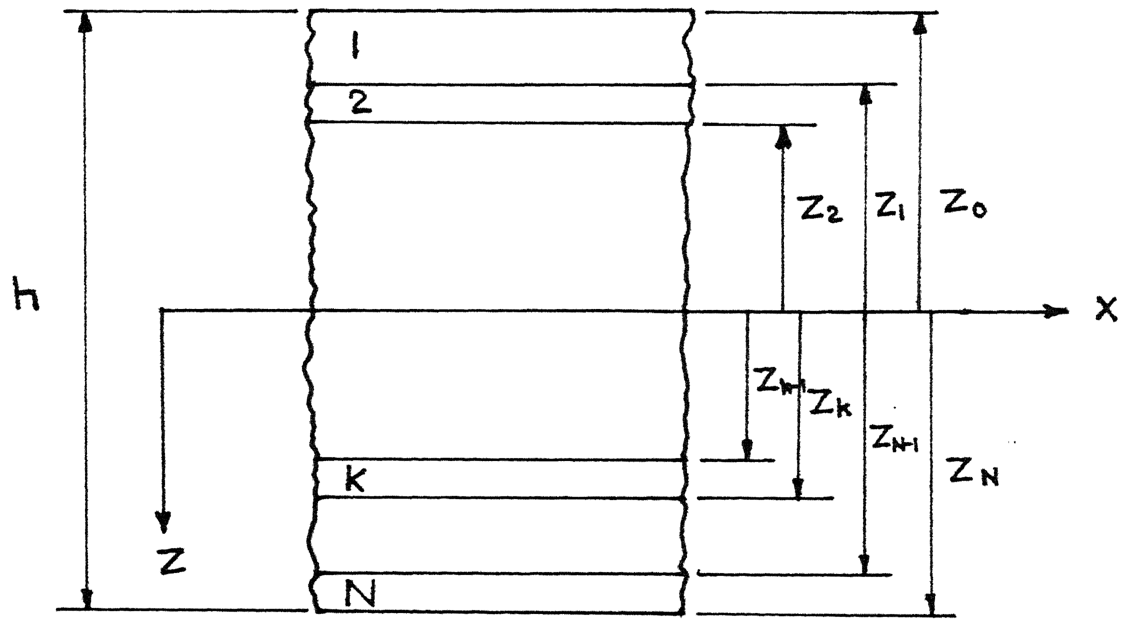


Fig. 2·3 .

where $\bar{Q}_{11}$, $\bar{Q}_{12}$, $\bar{Q}_{16}$ are as given in APPENDIX-I.

Aggregate of the forces and moments for N-layered laminate is

$$(N_x, M_x) = \sum_{K=1}^{N} \int_{Z_{k-1}}^{Z_k} \sigma_x \, (1, Z) \, dZ \tag{2.3}$$

where Z_k, Z_{k-1} are as shown in Fig. 2.3. Substituting the value of σ_x we get,

$$N_x = A_{11} \frac{\partial u_o}{\partial x} - B_{11} \frac{\partial^2 w}{\partial x^2} \qquad \text{(a)}$$

$$(2.4)$$

$$M_x = B_{11} \frac{\partial u_o}{\partial x} - D_{11} \frac{\partial^2 w}{\partial x^2} \qquad \text{(b)}$$

where,

$$\left[A_{11}, B_{11}, D_{11}\right] = \sum_{K=1}^{N} \bar{Q}_{11}^{k} \left[(Z_k - Z_{k-1}),\ (Z_k^2 - Z_{k-1}^2),\ (Z_k^3 - Z_{k-1}^3)\right] \qquad \text{(c)}$$

The equation for vibration of the composite beam is given by

$$\frac{\partial^2 M(x,t)}{\partial x^2} + \rho(x) \frac{\partial^2 w(x,t)}{\partial t^2} + f(x,t) = 0 \tag{2.5}$$

Substituting for M(x,t) we get,

$$\frac{\partial^2}{\partial x^2}\left[D(x)\,\frac{\partial^2 w(x,t)}{\partial x^2}\right] + \rho(x)\,\frac{\partial^2 w(x,t)}{\partial t^2} + f(x,t) = 0 \qquad \text{(a)}$$

(2.6)

where,

$$D(x) = \frac{B_{11}^2}{A_{11}} - D_{11} \qquad \text{(b)}$$

Finite Element Analysis of the Beam

A linear finite element with two nodes is taken with shape functions,

$$N_1 = 1 - \frac{3x^2}{e^2} + \frac{2x^3}{e^3}$$

$$N_2 = x - \frac{2x^2}{e} + \frac{x^3}{e^2}$$

$$N_3 = \frac{3x^2}{e^2} - \frac{2x^3}{e^3} \qquad (2.7)$$

$$N_4 = -\frac{x^2}{e} + \frac{x^3}{e^2}$$

where,

x - local co-ordinate of the element.

e - length of the element.

If $w(x,t)$ is deflection at any point x and at time t as shown in Fig. 2.4 we have,

$$w = \lfloor N \rfloor \{w_i\} , \qquad (2.8)$$

where,

$$\lfloor N \rfloor = \lfloor N_1 , N_2 , N_3 , N_4 \rfloor$$

$$w_i = \text{deflection at node } i$$

Applying Galerkin method to equation (2.6a), we get,

$$\int_0^e (\lfloor N'' \rfloor^T \lfloor N'' \rfloor \cdot D(x) - \alpha^2 \rho(x) \lfloor N \rfloor^T \lfloor N \rfloor) \, dx \cdot \{w_i\}$$

$$= \lfloor N' \rfloor^T \left[D(x) \cdot \frac{\partial^2 w}{\partial x^2} \right] \Big|_0^e - \lfloor N \rfloor^T \frac{\partial}{\partial x} \left[D(x) \frac{\partial^2 w}{\partial x^2} \right] \Big|_0^e$$

$$- \int_0^e \lfloor N \rfloor^T f(x,t) \, dx \qquad (2.9)$$

Above equation can be written for an element as,

$$([K] - \alpha^2 [M]) \{w_i\} = \{F_i\} + \begin{bmatrix} -Q_1 \\ -M_1 \\ +Q_2 \\ +M_2 \end{bmatrix} \qquad (2.10)$$

The elemental stiffness and mass matrices are assembled and for free vibration case we get,

$$([K] - \alpha^2 [M]) \{w_i\} = 0 \qquad (2.11)$$

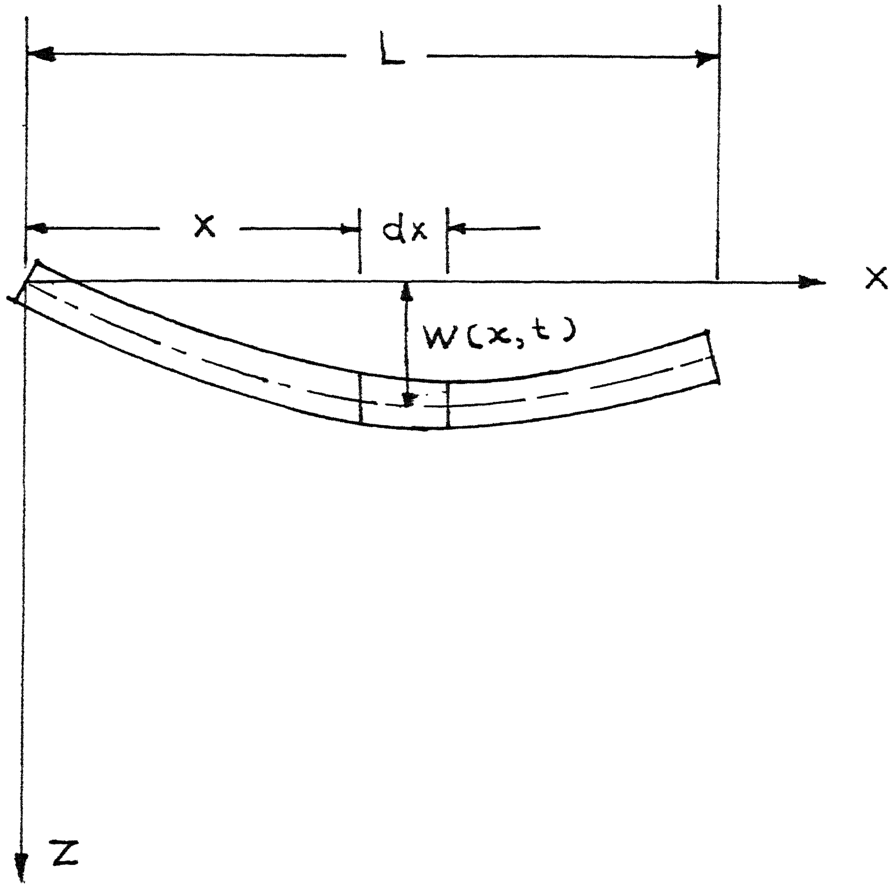


Fig. 2·4 .

where, $[K]$ and $[M]$ are global stiffness and mas matrices. The equation (2.11) is solved using standard algorithm. The values of α are the natural frequencies for the beam.

If $\bar{\omega}$ is the forcing frequency of the sinusoidal load then we get,

$$([K] - \bar{\omega}^2 [M]) \{\bar{q}\} = \{\bar{F}\} \tag{2.12}$$

where,

$\bar{q}$ - displacement at node

$\bar{F}$ - nodal force vector

Solution of equation (2.12) gives the maximum displacement at each node.

The boundary conditions considered are as follows:

(a) Simply supported

$$w = 0 \quad \text{at } x = 0 \text{ and } x = L. \tag{2.13}$$

(b) Fixed-free

$$w = 0 \text{ at } x = 0 \text{ and } w' = 0 \text{ at } x = 0 \tag{2.14}$$

(c) Fixed-fixed

$$w = 0 \text{ and } w' = 0 \text{ at } x = 0 \text{ and } x = L \text{ respectively.} \tag{2.15}$$

LAMINATED AXISYMMETRIC SHELLS

A general case of axisymmetric shell is shown in in Fig. 2.5. Global co-ordinates Θ, Y, R are shown with respect to the shell. The elements are shown on the cross-section of the shell with the respective nodes. This is a parabolic curved element with three nodes per element and variable thickness at each node. The local co-ordinates for each element are given by ξ , η , ζ as shown in the figure.

Considering a single element, the geometric shape functions are,

$$N_1 = \frac{1}{2}(-\eta + \eta^2)$$

$$N_2 = 1 - \eta^2 \qquad (2.16)$$

$$N_3 = \frac{1}{2}(\eta + \eta^2)$$

The global and local co-ordinates for this element are related as follows:

$$\Theta = \xi \qquad (a)$$

$$(2.17)$$

$$\begin{Bmatrix} Y \\ R \end{Bmatrix} = \Sigma N_i \begin{Bmatrix} Y_i \\ R_i \end{Bmatrix} + \Sigma N_i \frac{t_i}{2} \zeta \bar{V}_{3i} \qquad (b)$$

where,

Y_i, R_i - global co-ordinates at node i

t_i - the thickness of the shell at the i^{th} node

$\bar{V}_{3i}$ - unit vector at node i in the direction normal to the middle surface

i - 1,2,3 (nodes)

At any point in the shell along ξ, η, ζ, $\bar{V}_1$, $\bar{V}_2$, $\bar{V}_3$ are unit vectors. Let $\bar{i}$, $\bar{j}$, $\bar{k}$ be unit vectors along Θ, Y, R, Thus

$$\bar{V}_i = \bar{i}$$

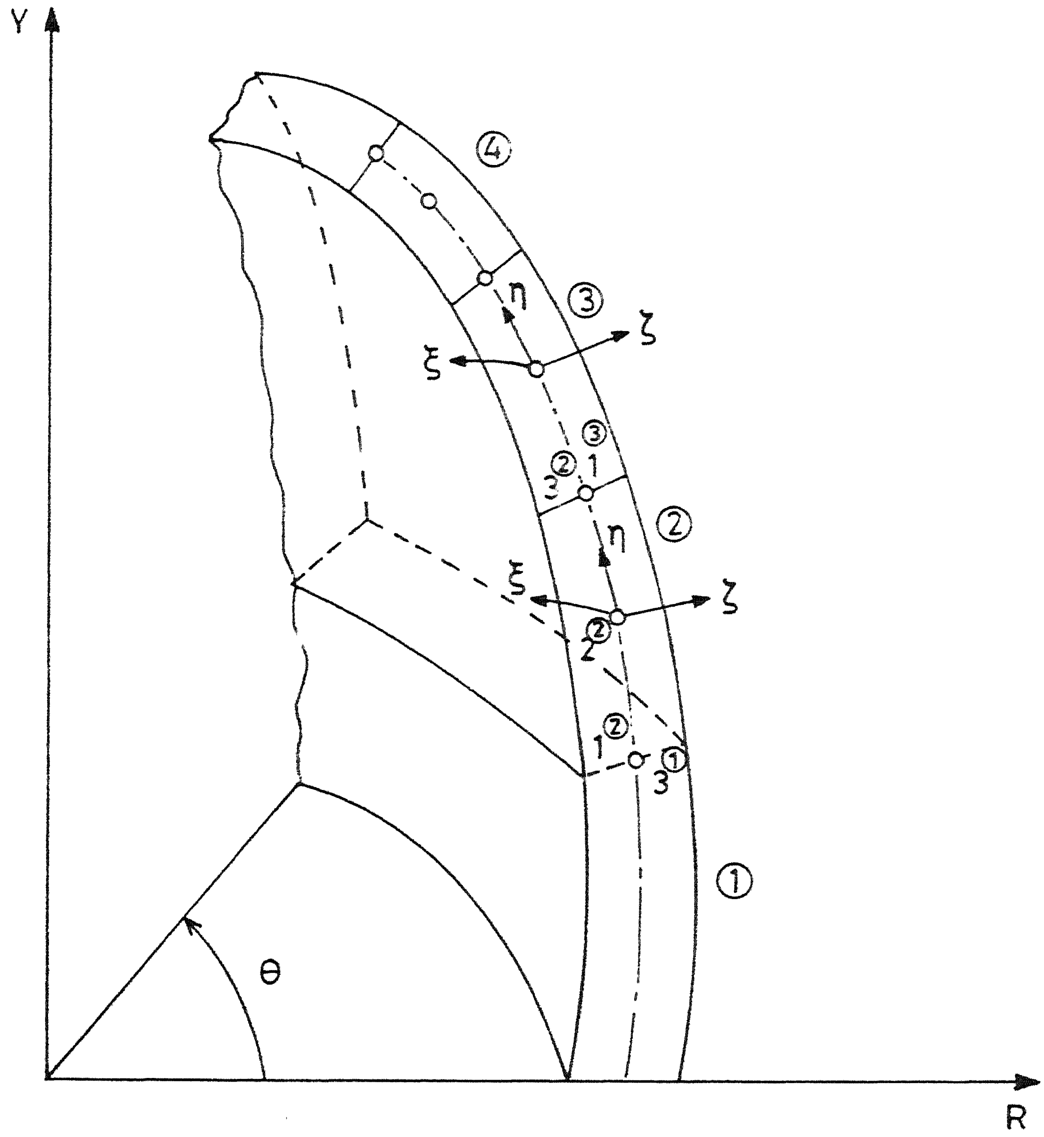
$$\bar{V}_2 = (\bar{j} \cdot \frac{\partial Y}{\partial \eta} + \bar{k} \cdot \frac{\partial R}{\partial \eta}) / \sqrt{|(\partial Y/\partial \eta)^2 + (\partial R/\partial \eta)^2|}$$

$$V_3 = (-\bar{j} \cdot \partial R/\partial \eta + \bar{k} . \partial Y/\partial \eta)/\sqrt{|(\partial Y/\partial \eta)^2 + (\partial R/\partial \eta)^2|} \tag{2.18}$$

The unit vectors at node i are found by dropping the second term of right hand side of equation (2.17b).

DISPLACEMENT FUNCTIONS

At any node i the displacements are u_i, v_i, w_i and the normal rotation α_i in the Y-R plane as shown in Fig. 2.6.



1 , ----- 4 - Element Number
1 , 2 , 3 - Node Number
θ , Y , R - Global Co-ordinates
ξ , ζ , η - Local Co-ordinates

Fig. 2.5 Global and local co-ordinate system.

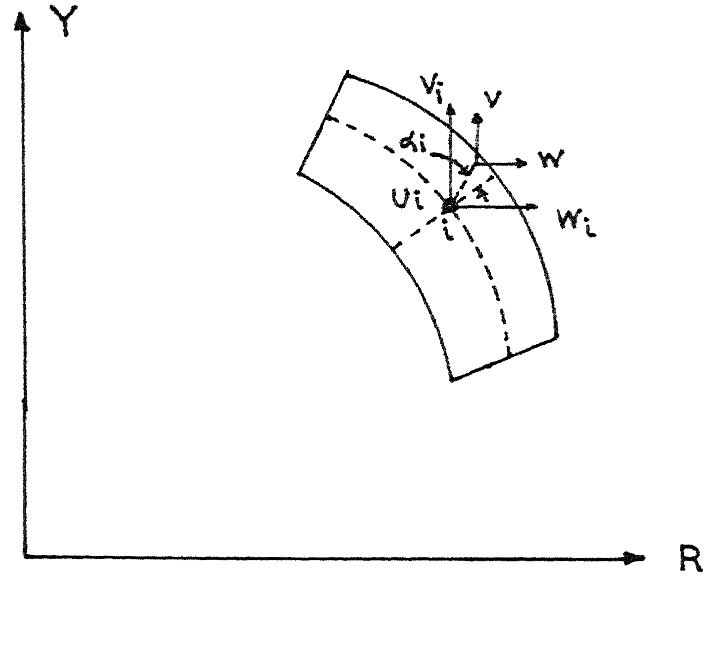


Fig. 2·6.

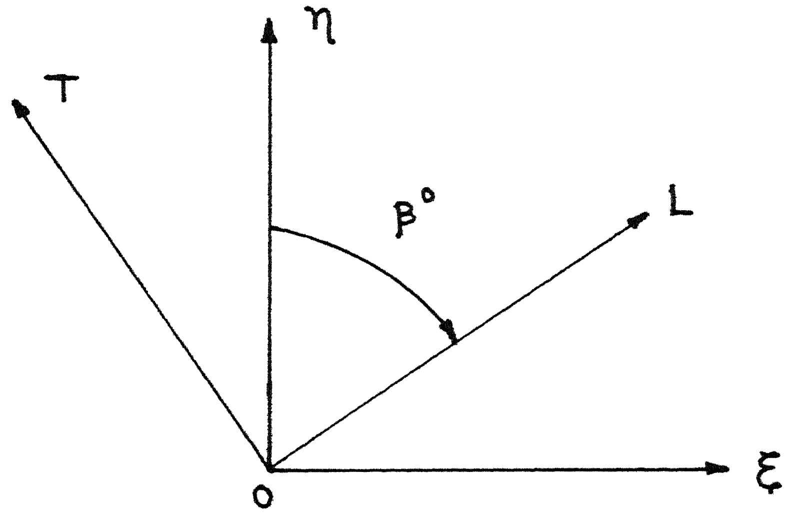


Fig. 2·7

At any point on the shell the displacement u,v,w are given by,

$$u = \Sigma N_i \; u_i$$
$$v = \Sigma N_i \; v_i + \Sigma N_i \frac{t_i}{2} \zeta (\bar{V}^Y_{2i} \alpha_i + \bar{V}_{3i}) \qquad (2.19)$$
$$w = \Sigma N_i \; w_i + \Sigma N_i \frac{t_i}{2} \zeta (\bar{V}^R_{2i} \alpha_i + \bar{V}_{3i})$$

where,

$i = 1,2,3.$

$\bar{V}^Y_{2i}$, $\bar{V}^R_{2i}$ - components of unit vector $\bar{V}_2$ at node i along Y and R axes.

Let,

$$\{\delta_i\} = \lfloor u_i, v_i, w_i, \alpha_i \rfloor^T \qquad (a)$$

For node, i = 1,2,3,. Therefore,

$$\{\delta\} = \lfloor \delta_1, \delta_2, \delta_3 \rfloor^T \qquad (b) \qquad (2.20)$$

$$\lfloor u, v, w \rfloor^T = [N] \; \{\delta\} \qquad (c)$$

where $[N]$ is matrix of shape functions.

STRESS-STRAIN RELATIONS

The three dimensional strains with respect to global

axes are given by,

$$\begin{bmatrix} \varepsilon_\Theta \\ \varepsilon_Y \\ \varepsilon_R \\ \gamma_{\Theta Y} \\ \gamma_{\Theta R} \\ \gamma_{YR} \end{bmatrix} = \begin{bmatrix} \frac{w}{R} + \frac{1}{R}\frac{\partial u}{\partial \Theta} \\ \partial v/\partial Y \\ \partial w/\partial R \\ \frac{1}{R}\frac{\partial v}{\partial \Theta} + \frac{\partial u}{\partial Y} \\ \frac{1}{R}\frac{\partial w}{\partial \Theta} + \frac{\partial u}{\partial R} - \frac{u}{R} \\ \frac{\partial w}{\partial Y} + \frac{\partial v}{\partial R} \end{bmatrix} \qquad (2.21)$$

The global derivatives are obtained as follows:

$$\frac{\partial N_i}{\partial \Theta} = 0 \qquad (a)$$

$$\frac{\partial N_i}{\partial Y} = \frac{\partial R}{\partial \zeta} \cdot \frac{\partial N_i}{\partial \eta} \;/\; |J| \qquad (b) \qquad (2.22)$$

$$\frac{\partial N_i}{\partial R} = -\frac{\partial Y}{\partial \zeta} \cdot \frac{\partial N_i}{\partial \eta} \;/\; |J| \qquad (c)$$

where,

$$|J| = \frac{\partial Y}{\partial \eta} \cdot \frac{\partial R}{\partial \eta} - \frac{\partial Y}{\partial \zeta} \cdot \frac{\partial R}{\partial \eta} \qquad (d)$$

The strains with respect to the local axes are obtained from the strains along the global directions. Finally we get strains along the local axes in terms of displacements at nodal points as follows:

$$\left[\varepsilon_\xi \;\; \varepsilon_\eta \;\; \varepsilon_\zeta \;\; \gamma_{\xi\eta} \;\; \gamma_{\xi\zeta} \;\; \gamma_{\eta\zeta} \right]^T = \left[B \right] \{\delta\} \tag{2.23}$$

Let β be the orientation of the fibres in a layer as shown in Fig. 2.7. Therefore, the transformation matrix is given as

$$\left[T \right] = \begin{bmatrix} m^2 & n^2 & 0 & -2mn & 0 & 0 \\ n^2 & m^2 & 0 & 2mn & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ mn & -mn & 0 & m^2-n^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & m & -n \\ 0 & 0 & 0 & 0 & n & m \end{bmatrix} \tag{2.24}$$

where,

$$m = \sin\beta \; , \quad n = \cos\beta$$

The stress-strain relation with respect to elastic

axes is given by,

$$\begin{bmatrix} \varepsilon_L \\ \varepsilon_T \\ \varepsilon_\zeta \\ \gamma_{LT} \\ \gamma_{L\zeta} \\ \gamma_{T\zeta} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_L \\ \sigma_T \\ \sigma_\zeta \\ \tau_{LT} \\ \tau_{L\zeta} \\ \tau_{T\zeta} \end{bmatrix} \qquad \text{(a)} \quad (2.25)$$

where,

L - Longitudinal axis of fibres.

T - Transverse axis of fibres.

$$S_{11} = \frac{1}{E_L}, \quad S_{22} = S_{33} = \frac{1}{E_T}$$

$$S_{44} = \frac{1}{G_{LT}}, \quad S_{55} = S_{66} = \frac{1}{G_{TT}} \qquad \text{(b)}$$

$$S_{12} = S_{13} - \frac{\nu_{LT}}{E_T}$$

$$S_{23} = - \frac{\nu_{TT}}{E_T}$$

in which E, G, ν are used for modulus of elasticity,

modulus of rigidity and Poisson's ratio.

The stress-strain relation with respect to local axes are given by

$$\left[\sigma_\xi \quad \sigma_\eta \quad \sigma_\zeta \quad \tau_{\xi\eta} \quad \tau_{\xi\zeta} \quad \tau_{\eta\zeta}\right]^T = [T] \quad [S]^{-1} \quad [T]^T \left[\varepsilon_\xi \quad \varepsilon_\eta \quad \varepsilon_\zeta \quad \gamma_{\xi\eta} \gamma_{\xi\zeta} \gamma_{\eta\zeta}\right] \tag{2.26}$$

The element stiffness matrix $[K]$ is given by,

$$\begin{aligned} [K] &= \iiint [B]^T \quad [T] \quad [S]^{-1} \quad [T]^T \quad [B] \; d\,(vol) \\ &= \int_{-1}^{1} \int_{-1}^{1} [B]^T \quad [D] \quad [B] \; 2\pi R \quad |J| \; d\eta \quad d\zeta \end{aligned} \tag{2.27}$$

where,

$|J|$ = The Jacobian given by equation (2.22d)

$[D] = [T] \; [S]^{-1} \; [T]^T$

As $[D]$ matrix is different for each layer, the variable ζ in the K^{th} layer is changed to ζ_k such that ζ_k varies from -1 to 1 in the K^{th} layer. Then,

$$\zeta = -1 + \frac{1}{t} \left[-h_k \; (1 - \zeta_k) + 2 \sum_{j=1}^{K} h_j \right] \tag{2.28}$$

$$d\zeta = \frac{h_k}{t} \quad d\zeta_k$$

where h_k is the thickness of K^{th} layer. With this substitution we get,

$$[K] = \sum_{k=1}^{n} \left(\int_{-1}^{1} \int_{-1}^{1} [B]^T [D]_k [B] \; 2\pi R \, |J| \, \frac{h_k}{t} \, d\eta \, d\zeta_k \right)$$

(2.29)

where n is total number of layers.

Equation (2.29) is integrated numerically using a three point Gauss method.

The elemental mass matrix is obtained as,

$$[M] = \sum_{k=1}^{n} \int_{vol} [N]^T [\rho_c]_k [N] \; d\,(vol)$$

$$= \sum_{k=1}^{n} \left(\int_{-1}^{1} \int_{-1}^{1} [N]^T [\rho_c]_k [N] \, 2\pi R |J| \; \frac{h_k}{t} \, d\eta \, d\zeta_k \right)$$

(2.30)

where, $[\rho_c]_k$ is the density of the composite in layer k.

The element stiffness and mass matrices are assembled to get global stiffness and mass matrices. They are used to solve the generalised eigenvalue problem.

$$[k] \{\delta\} = \omega^2 [M] \{\delta\} \tag{2.31}$$

The values of ω give natural frequencies and the values

of $\{\delta\}$ give mode shapes. Equation (2.31) is solved using standard eigen value solution subroutine.

The sinusoidal forced vibration response is obtained as given by equation (2.12).

The two types of boundary conditions considered for the axisymmetric shells are as follows:

(a) Simply supported without axial constraint

$u=0$, $w=0$, $\alpha=0$ and $v \neq 0$ at both ends.

These boundary conditions can be approximated by attaching thin, flat, circular diaphragms to the ends of the cylinder. The diaphragms have considerable stiffness in their own plane, thereby restraining the u and w components of the shell displacement at their mutual boundaries. However, the diaphragms, as a result of their thinness, generate negligible bending moment and longitudinal force in the shell as the shell deforms and do not restrain axial deformation, v.

(b) Simply supported with axial constraint

$u=0$, $v=0$, $w=0$ and $\alpha=0$ at both ends.

COMPARSION WITH OTHER METHODS OF ANALYSES:

Laminated Beams

Most of the work is done on the orthotropic beams but on anisotropic laminated FRP beams, work is rather scarce. Singh [26] used laminate composite theory with finite difference method.

In the present work, the laminate theory is used with the following assumptions.

1. The normal to the middle surface remains normal to the surface after deformation.
2. The stretching along this normal is negligible and hence taken to be zero. This means that there is no strain along the thickness direction.
3. Bending deformation is in a plane and hence displacement along width is zero and displacement along thickness and length are constant across the width.
4. Plane sections remain plane during the deformation.

The present model does not consider shear deformation and bending-torsion coupling.

Axisymmetric Shells

Most of the work on vibrations of laminated fibre reinforced composite shells of revolution considers the circumferential mode frequency analysis. Therefore either triangular or quadrilateral elements have been used. This led to consideration of large number of degrees of freedom per node to simulate adequate inter-element

compatibility. If these elements are used for axisymmetric mode frequency analysis, the size of stiffness matrix becomes very large.

Conical and cylindrical elements were used with Fourier distribution of loads and displacements along the circumferential direction. These elements are useful for getting both types of frequencies for cylindrical and conical shells. But these elements involve approximations when used for general axisymmetric shells.

The use of parabolic curved shell element in the present analysis is suitable for general axisymmetric shells. Some of its features are:

1. The approximation for the surface is not required.
2. The axisymmetric thickness variation along the length of shell is permitted.
3. Variation of density and elastic coefficients for each lamina in the laminate is permitted . Due to the use of an isoparametric element following additional features arise.
4. Distribution of displacement is non-linear across the thickness and that of strain is linear.
5. Due to lamination of orthotropic layers, anisotropic properties are present across the thickness. Therefore, a refinement is made from other theories in assuming that the normal to the middle surface after deformation does not remain straight.
6. Strain in the direction normal to the mid-surface is considered.

CHAPTER III

RESULTS AND DISCUSSION

The software developed has five different modules. The execution of the main program "PROJECT" will lead to the MENU. In this MENU, all modules are connected as shown in Fig. 3.1. Selection of a particular module can be done with the help of a cursor. Functionally the modules can be catagorised in the following way.

(I) Laminated Beams

There are two modules for the analysis of laminated beams.

(a) Vibration analysis of laminated beams

The vibration analysis can be done for two cases;

(i) Analysis of Natural frequencies and mode shapes.

(ii) Analysis of response for the sinusoidal forced vibration.

Module "FVBEAM" is for the vibration analysis of the laminated composite beams. It works in the following different phases.

(1) The input data is collected from the user on the terminal. This data is also written in the file "DAT-INP".

(2) The processing of data is done and the coefficients A_{11}, B_{11},D_{11} from equation (2.4) and D(x) from equation (2.6 b) are calculated.

(3) Elemental mass and stiffness matrices are calculated. These are then assembled according to the boundry conditions to form global mass and stiffness matrices. The equation

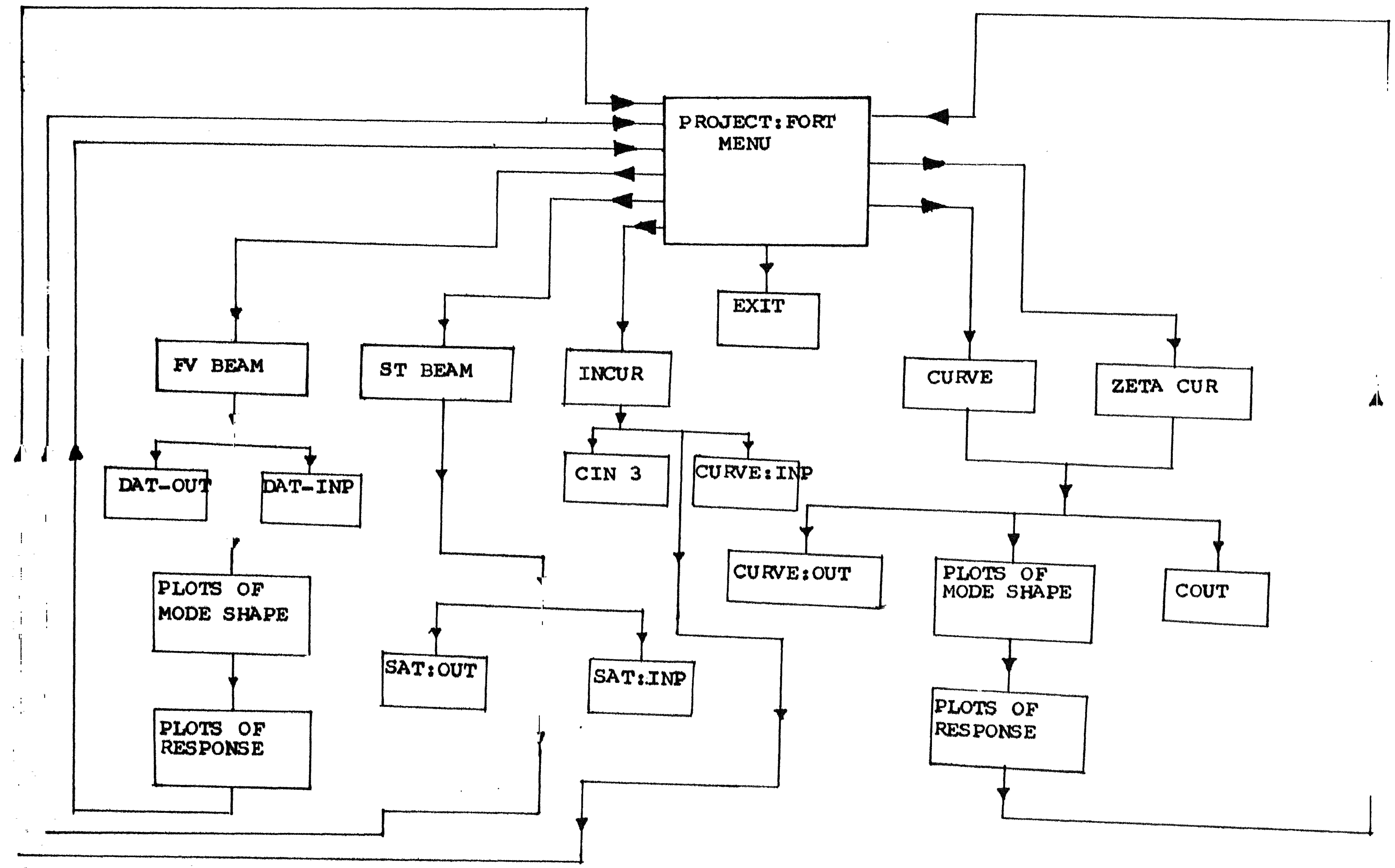

PROJECT:FORT
MENU
EXIT
FV BEAM
ST BEAM
INCUR
CURVE
ZETA CUR
DAT-OUT
DAT-INP
CIN 3
CURVE:INP
CURVE:OUT
PLOTS OF
MODE SHAPE
COUT
SAT:OUT
SAT:INP
PLOTS OF
RESPONSE
PLOTS OF
MODE SHAPE
PLOTS OF
RESPONSE

Fig. 3.1

(2.11) is solved using NAG subroutine (F02AEF). The results i.e. natural frequencies and modal vectors are stored in the file "DAT-OUT".

(4) For getting the sinusoidal forced vibration response, first a file "DATA:INP" is to be created in which the forcing frequency is written first and then the nodal values of force are written. The response vectors are written in file "DAT-OUT".

(5) Plots of mode shapes and response are then drawn on the workstation.

(b) Static load analysis of laminated beams

"STBEAM" is for static load analysis. The execution of this program will give the strains and stresses in each layer at a particular section of the beam. Different types of loading conditions are provided. This program is based on the exact analysis. Execution of "STBEAM" creates "SAT:INP" and "SAT:OUT" files.

(II) Axisymmetric Shells

There are three modules for the analysis of laminated axisymmetric shells.

(a) Execution of "INCUR" will lead to an interactive session with the user. User has to feed data on the terminal. This data consists of following items:

(i) Element node point co-ordinates in Y-R plane as shown in Fig. 2.5. Thickness at each node.

(ii) Material properties.

(iii) Number of laminae, orientation and thickness of the laminae.

(iv) Forcing frequency of the sinusoidal force and the nodal force values. These values will depend on the type of loading e.g. concentrated load, uniformely distributed load etc.

(b) Module "CURVE" is for the vibration analysis of laminated composite axisymmetric shells.

(c) Module "ZETACUR" is for the vibration analysis of laminated composite axisymmetric shells in which the strain along the direction normal to the middle plane is also considered.

After execution of any of these two modules i.e. "CURVE" and "ZETACUR" following operations are performed;

(1) Elemental stiffness and mass matrices are calculated using equations (2.29) and (2.30).

(2) Natural frequencies and modal vectors are obtained using the equation (2.31). Natural frequencies are written in file "CURVE:OUT". Modal vectors are written in file "COUT".

(3) The response is written in file "CURVE:OUT".

(4) The plots of mode shapes are drawn on the workstation.

(5) The plots of response in the direction O,Y,R are drawn separately on the workstation.

For the detail knowledge about the program please refer to Appendix II.

RESULTS FOR COMPOSITE BEAMS AND SHELLS

Laminated Composite Beam

The program "FVBEAM" was checked for the non-dimensional frequency parameter Ω_j, for the isotropic case. The non-dimensional frequency parameter Ω_j is related to the natural frequency as,

$$\omega_j = \Omega_j^2 / L^2 (EI/\rho)^{1/2} \qquad j = 1, 2, 3.....(modes)$$

In tables 3.1, 3.2, 3.3 a comparison is given for the results obtained from "FVBEAM" with exact results from reference [21]. The convergence of the eigen values with increase in the number of finite elements to the exact value can be seen from the tables.

TABLE 3.1 Comparison of eigenvalues (Ω_j) for Simply Supported Beam

Mode Number j	2-Elements	4-Elements	8-Elements	10-Elements	Exact
1	9.9086	9.8730	9.8822	9.8208	9.8696
2	43.8178	39.6347	39.4930	39.4830	39.4784
3	110.1396	90.4495	88.9408	88.8715	88.8264
4	200.7984	175.2713	158.5369	158.1729	157.9137
5	--	--	249.0268	247.7139	246.7401

TABLE 3.2 Comparison of the eigenvalues (Ω_j) for Fixed-Free beam

Mode Number j	2-Elements	4-Elements	8-Elements	10-Elements	Exact
1	3.5176	3.5183	3.5670	3.0995	3.5008
2	22.2214	22.0605	22.0388	22.0304	22.0345
3	75.1571	62.1751	61.7354	61.7198	61.6972
4	218.1380	122.6577	121.1729	121.0151	120.8990
5	—	—	201.0170	200.3607	199.8595

TABLE 3.3 Comparison of the eigen values (Ω_j) for Fixed-Fixed beam.

Mode Number j	4-Elements	8-Elements	10-Elements	Exact
1	22.4029	22.3768	22.3789	22.3733
2	62.2433	61.7122	61.6858	61.6728
3	123.4854	121.1911	121.0237	120.9034
4	233.6242	201.1068	200.3897	199.8594
5	386.3797	302.4677	300.2653	298.5555

For high strength Graphite Epoxy Composite material following input data is taken.

$E_L = 1.2 \times 10^5$ N/mm^2		$\nu_{LT} = 0.27$	
$E_T = 1.0 \times 10^4$ N/mm^2		$L = 200$ mm	
$G_{LT} = 5.5 \times 10^4$ N/mm^2		$B = 30$ mm	
$\rho = 1.7 \times 10^{-9}$ N Sec^2 mm^{-4}		$h = 5$ mm	

For the laminated beams, $[0,90]_s$ and $[45,-45]_s$ composites are considered. Table 3.4, 3.5, 3.6 shows the values for the natural Frequencies in rad./sec. For three types of boundary conditions.

(a) Simply supported beam

(b) Fixed-Free beam

(c) Fixed-Fixed beam

The natural frequencies are plotted V_s the mode number for $[0,90]_s$ and $[45,-45]_s$ composites in Fig. 3.2, 3.3, 3.4. It is observed that the frequencies for the $[0,90]_s$ composite beam are higher for each mode than that of $[45,-45]_s$ composite beam.

TABLE 3.4 Natural Frequencies for Laminated Composite Beams.

(No. of Finite Elements = 10)

SIMPLY SUPPORTED BEAM

Mode	$[45,-45]_s$	$[0,90]_s$
1	469.26	518.82
2	1882.98	2064.13
3	4237.79	4643.07
4	7542.49	8263.49
5	11811.77	12941.03
6	17076.76	18709.32
7	23387.19	25263.18
8	30813.37	33759.29
9	39409.50	43177.22
10	52234.66	57228.49
11	61856.63	67770.48
12	75400.48	82609.19

TABLE 3.5 Natural Frequencies for Laminated Composite Beams.

(No. of Finite Elements = 10)

FIXED- FREE BEAM

MODE	$[45,-45]_s$	$[0,90]_s$
1	167.61	171.86
2	1050.00	1150.20
3	2942.85	3223.99
4	5770.67	6321.99
5	9554.02	10467.31
6	14313.03	15681.27
7	20081.92	22001.72
8	26904.18	29476.30
9	34786.72	38112.52
10	43244.93	47379.28
11	57557.07	63059.78
12	69536.14	76184.16

TABLE 3.6 Natural Frequencies for Laminated Composite Beams.

(No. of Finite Elements = 10)

FIXED-FIXED BEAM

MODE	$[45,-45]_s$	$[0,90]_s$
1	1066.88	1168.78
2	2941 60	3222.54
3	5770.75	6322.24
4	9555.19	10468.39
5	14371.62	15686.46
6	20096.54	22017.84
7	26943.22	29519.21
8	34875.34	38209.62
9	43384.16	47531.88
10	57886.75	63421.02
11	70232.94	76947.56
12	85485.66	93658.50

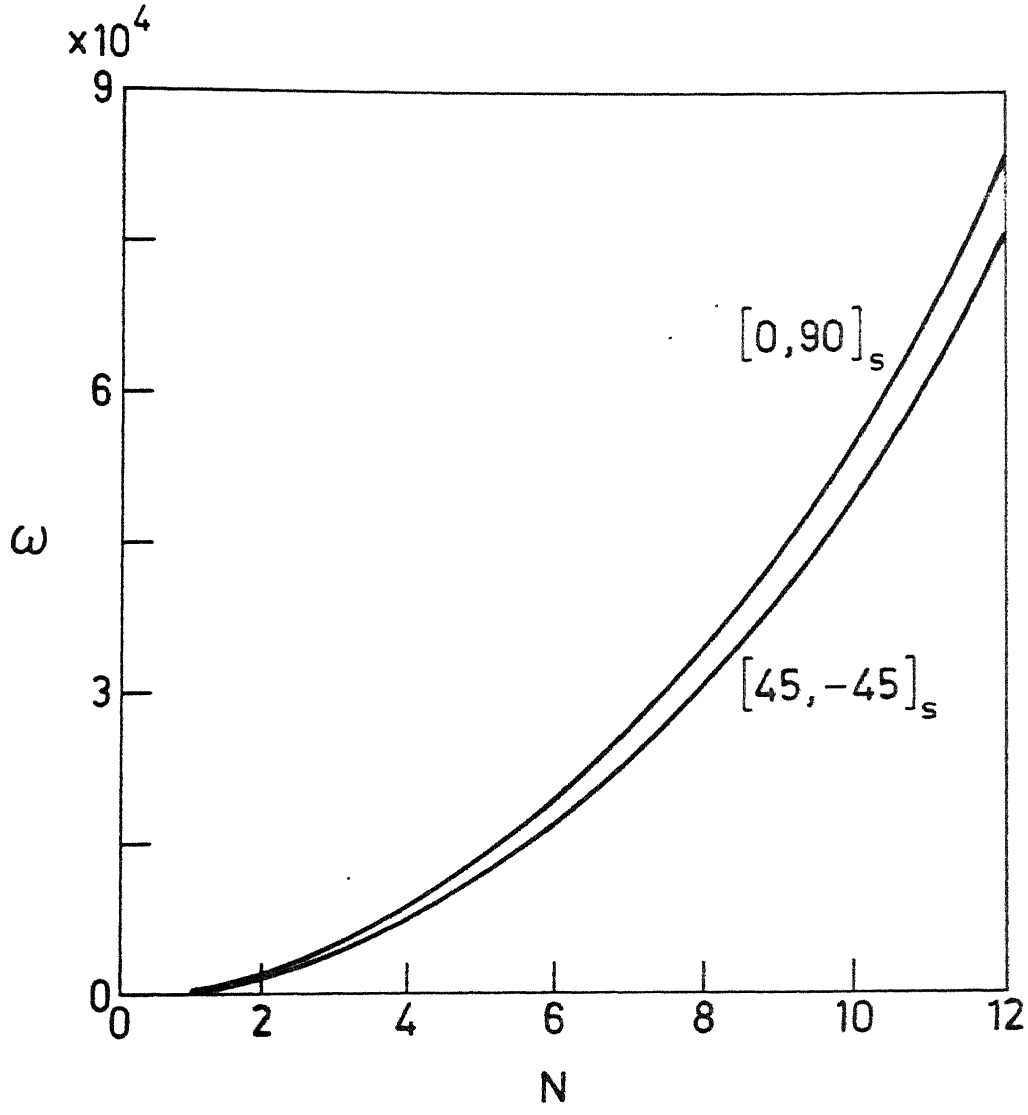


Fig. 3.2. Simply Supported Laminated Composite Beam.

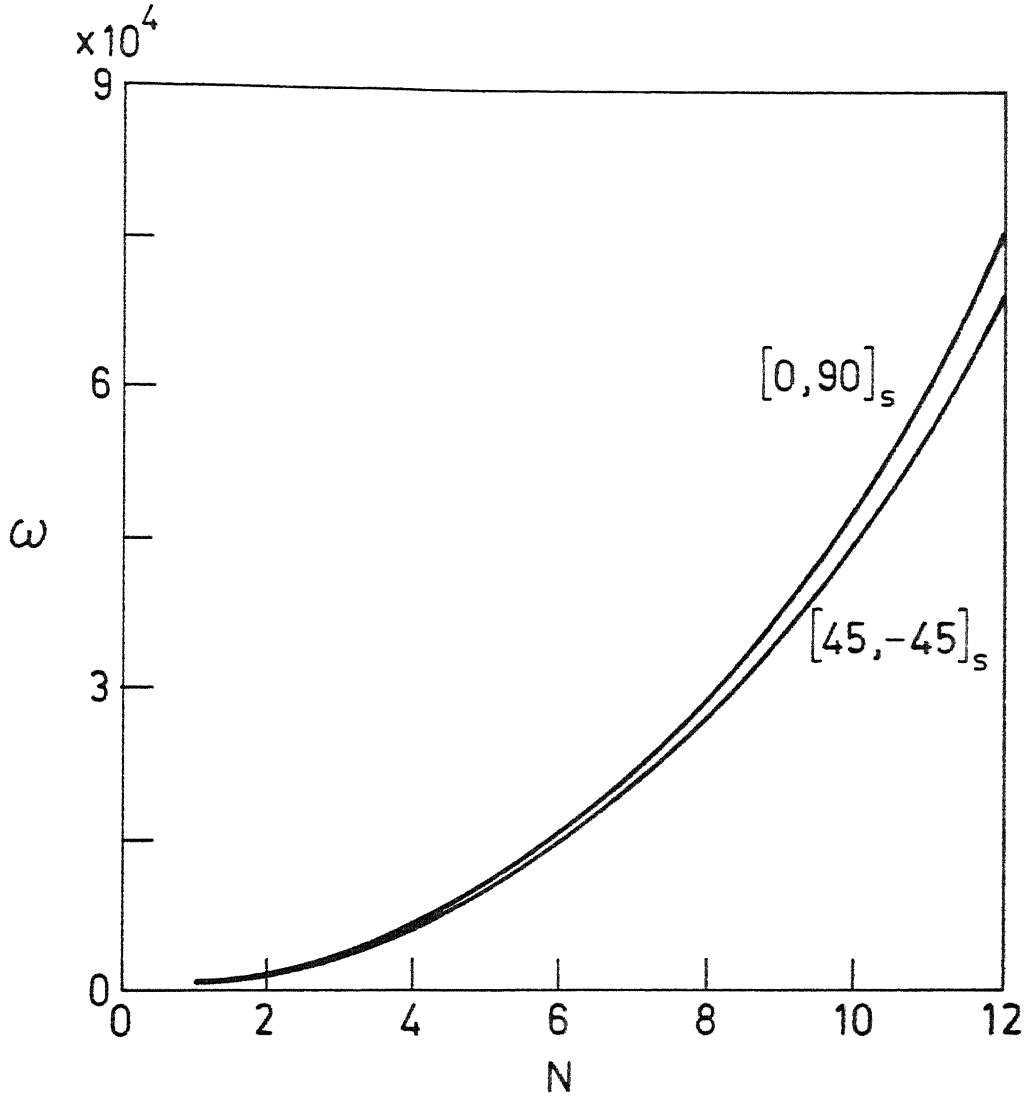


Fig. 3.3. Fixed-Free Beam of Laminated Composite.

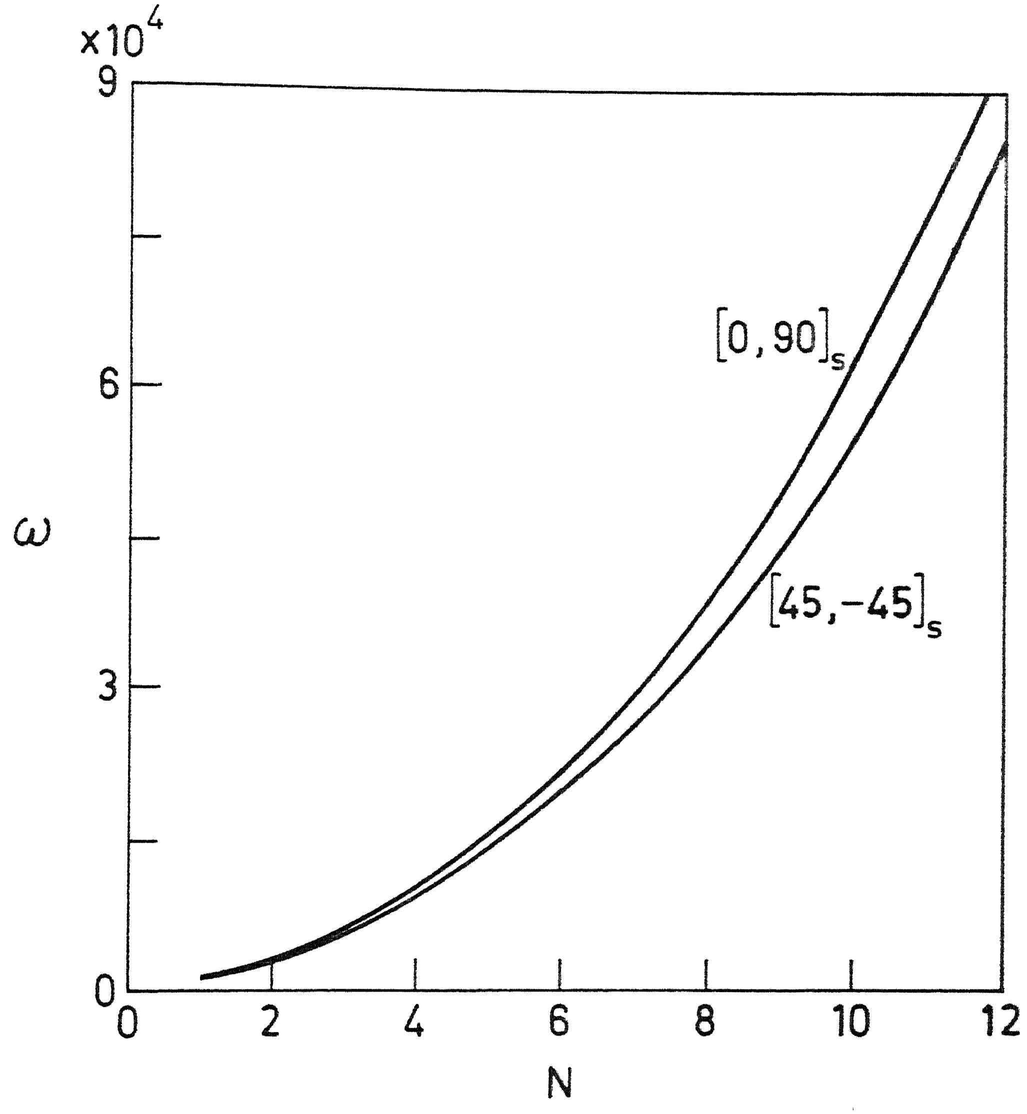


Fig. 3.4. Fixed-Fixed Beam of Laminated Composite.

Axisymmetric Shells of Revolution

In the program "CURVE", the equation (2.31) is solved using NAG subroutine (F02AEF) for getting the natural frequencies and modal vectors. This subroutine arranges frequenices in the ascending order. It is general observation (also ref.21) that the natural frequenices of shells do not fall in an ascending series with increasing values of the madal index. Therefore each frequency has to be checked for the modal shape. This will require lot of executions of "CURVE" module. The importance of the graphical plotting of the mode shapes is strongly felt while noting the natural frequencies. The numerical results are obtained from the program "CURVE". For high modules Graphite Epoxy composite, following data is

given as input.

E_L	$= 20 \times 10^6$ psi	$G_{TT'} = 0.5 \times 10^6$ psi	
E_T	$= 1 \times 10^6$ psi	$\nu_{LT} = 0.25$	
G_{LT}	$= 0.6 \times 10^6$ psi	$\nu_{TT} = 0.25$	

The non-dimensional frequencies are given by,

$\Omega = \omega\, h/\lambda_n \sqrt{\rho/E_L}$ where $\lambda_n = n\,\pi\, h/L$

n = Wave number (half)

L = Length of shell

h = thickness of cylinder

ρ = density of composite

ω = natural frequency (rad /sec)

R = radius of cylinder

For cylindrical shell, the plots for non-dimensional frequency Ω V_s $\lambda n/\pi$ are shown in Fig. 3.5, 3.6, 3.7, 3.8,. For two cases of R/h = 10 and R/h = 3 the results are taken for $[0,90]_s$ cross-ply laminated shell and $[45,-45]_s$ angle-ply laminated shell of the composite material.

Comparing the plots with those of reference [2], following observations can be made.

1. In Fig. 3.5 to Fig. 3.8, the plots are between Ω the non-dimensional frequency and $\lambda n/\pi$.

As $\lambda n/\pi$ ---> 0, h--> 0 but in the expression for Ω, $\Omega = \omega L/\pi \, n\sqrt{\rho/E_L}$. This means that as $\lambda n/\pi$ --> 0, Ω can have any value which is finite.

2. In the present case the variation of the natural frequency with the mode number n is approximately linear as observed from Fig. 3.9 and 3.10.

3. For a particular value of $\lambda n/\pi$ = constant, as the mode number n increases the ratio (h/L) decreases in the same proportion i.e.1/n x (h/L). This can be accomplished in two ways. One way is by decreasing the shell thickness h and keeping L constant. The other way is to keep the shell length L increasing and keeping h constant. In both ways the shell approaches to thin and long shell.

Presence of four degrees of freedom at each node leads to large stiffness coefficients for relative displacements along an edge corresponding to the shell thickness. This presents numerical problems and leads to the ill - conditioned equations when shell

thicknesses become small compared with the other dimensions in the element. Therefore in the present case, for thin, long shells the results are not reliable.

In the case of thick shells with R/h = 3 the results are comparable with those of reference [2]. For $\lambda n/\pi > 0.5$. In case of R/h = 10, the results are comparable with reference [2] for $\lambda n/\pi > 0.2$.

4. The results obtained from Flugge's theory are higher than that of refined theory of ref.[2]. Flugge's theory considers only inplane strains. The refined theory [2] considers the transverse normal strain and the effect of transverse shear deformation is included by introducing an approximate shear correction factor. The present mathematical model considers all three dimensional strains. The effect of anisotropy across the thickness, the non-linear distribution of displacement across the thickness and the linear distribution of strain across the thickness is considered. All these considerations lead to a refined model. Therefore, the non-dimensional frequencies obtained in the present case are smaller in magnitude.

From the analysis of the results it was observed that the natural frequencies change with the orientation of the fibres. A close observation at the plots of Fig. 3.11 leads to the following point.

The frequencies for the [0^o] composite are higher than that for the other orientation angles up to [45^o]. At the orientation of [45^o] the frequencies are minimum. As the

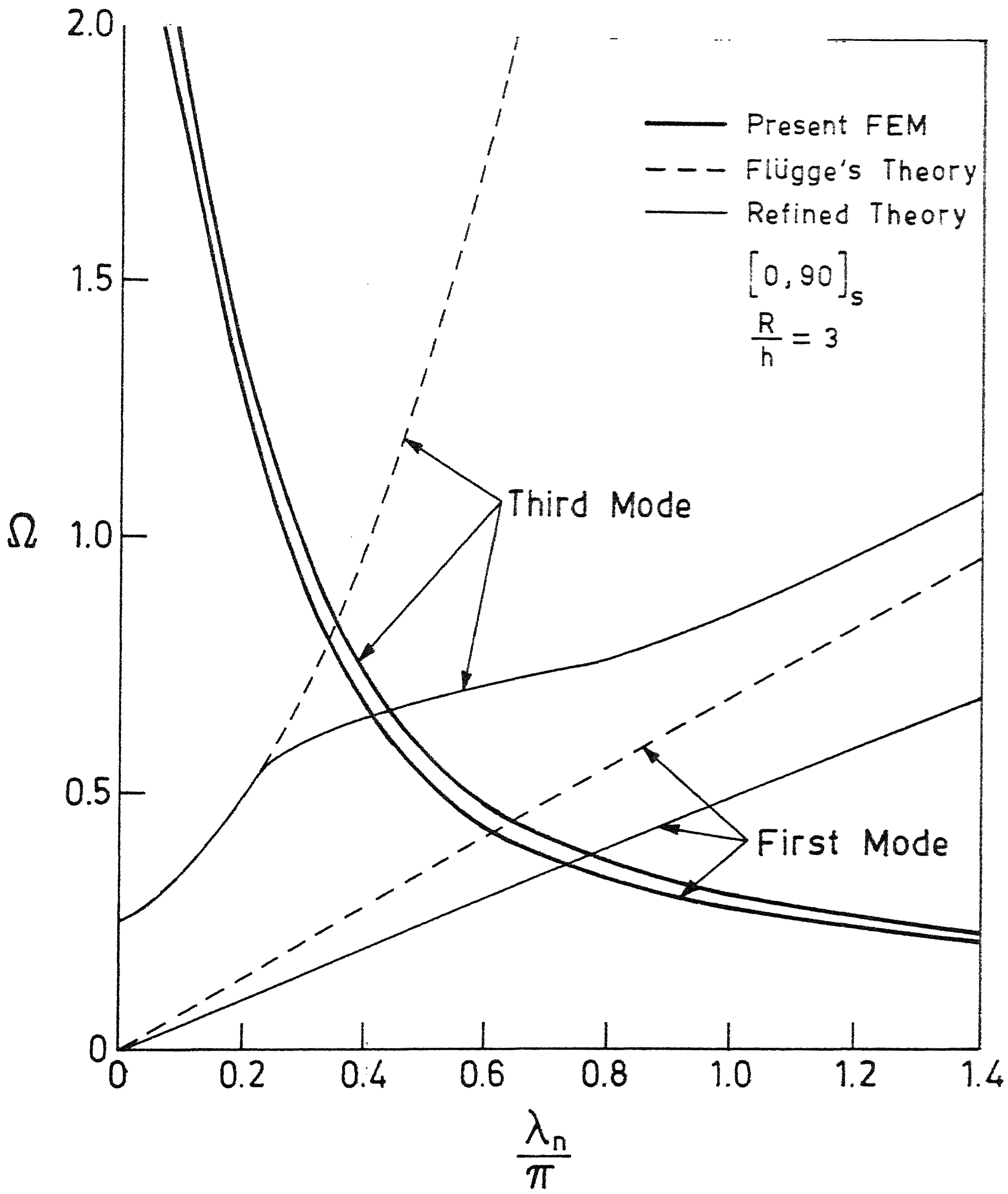


Fig. 3.5. Nondimensional Natural Frequency as a Function of h/l.

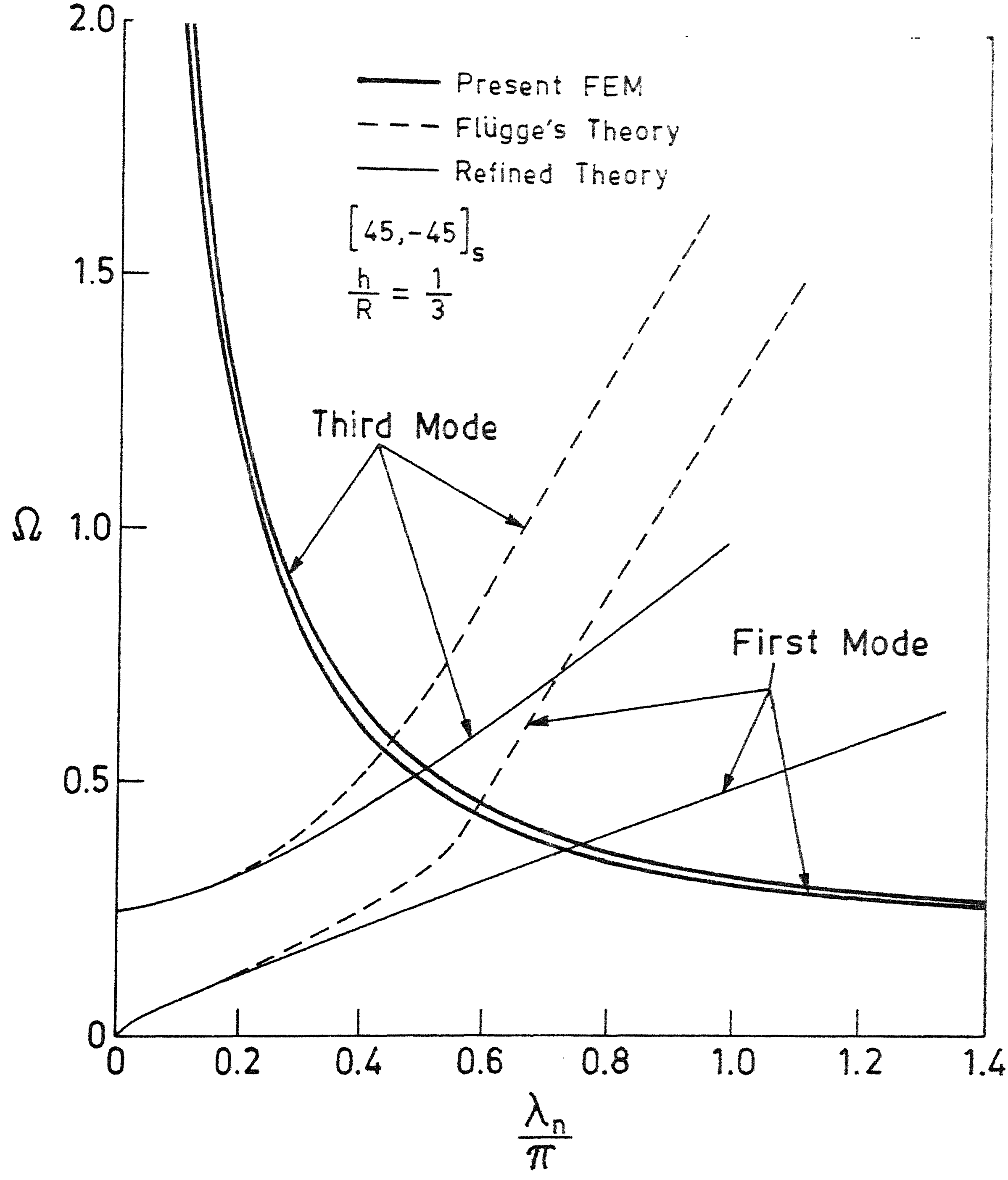


Fig. 3.6. Nondimensional Natural Frequency as a Function of h/l.

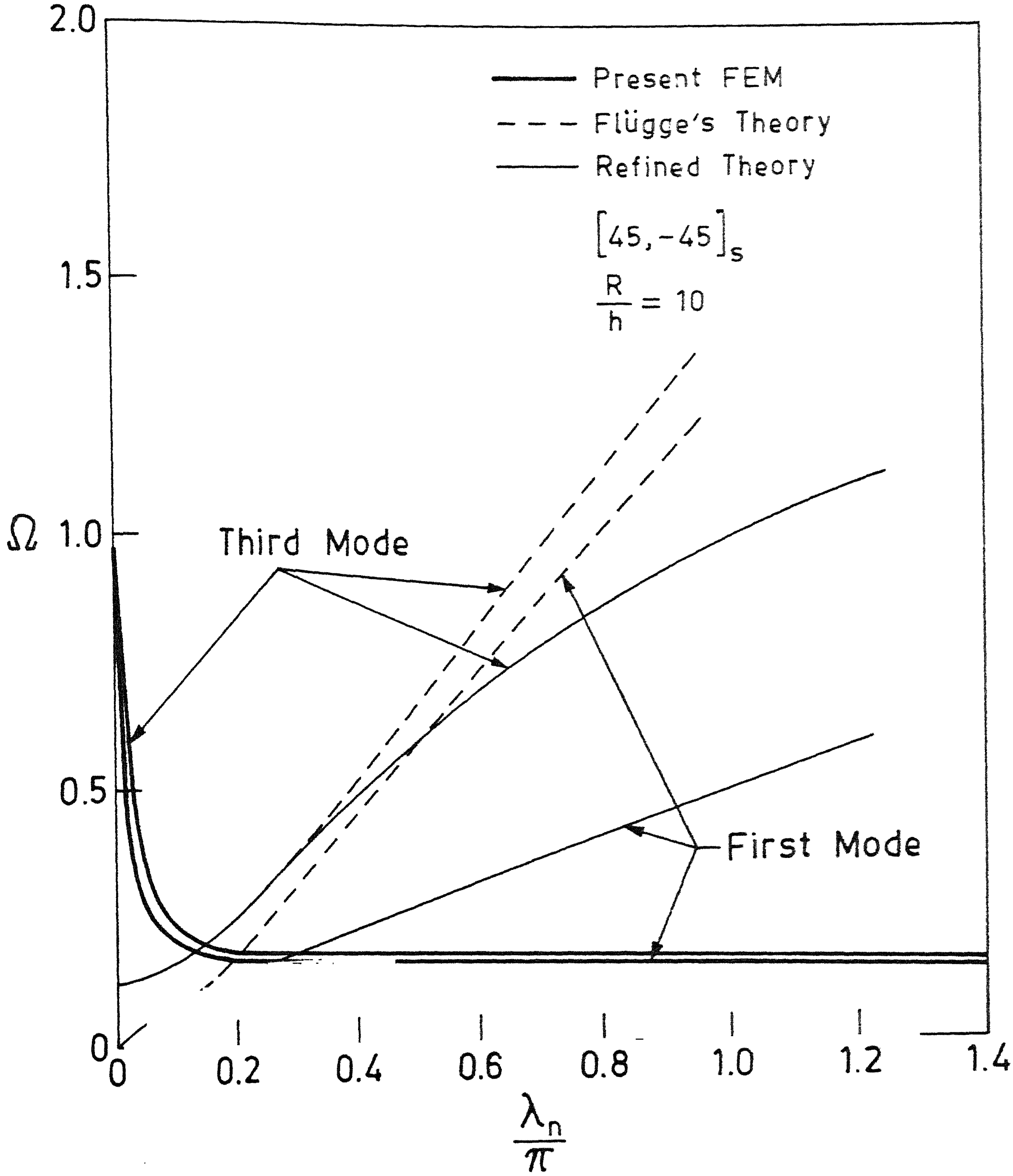


Fig. 3.7. Nondimensional Natural Frequency as a Function of h/l.

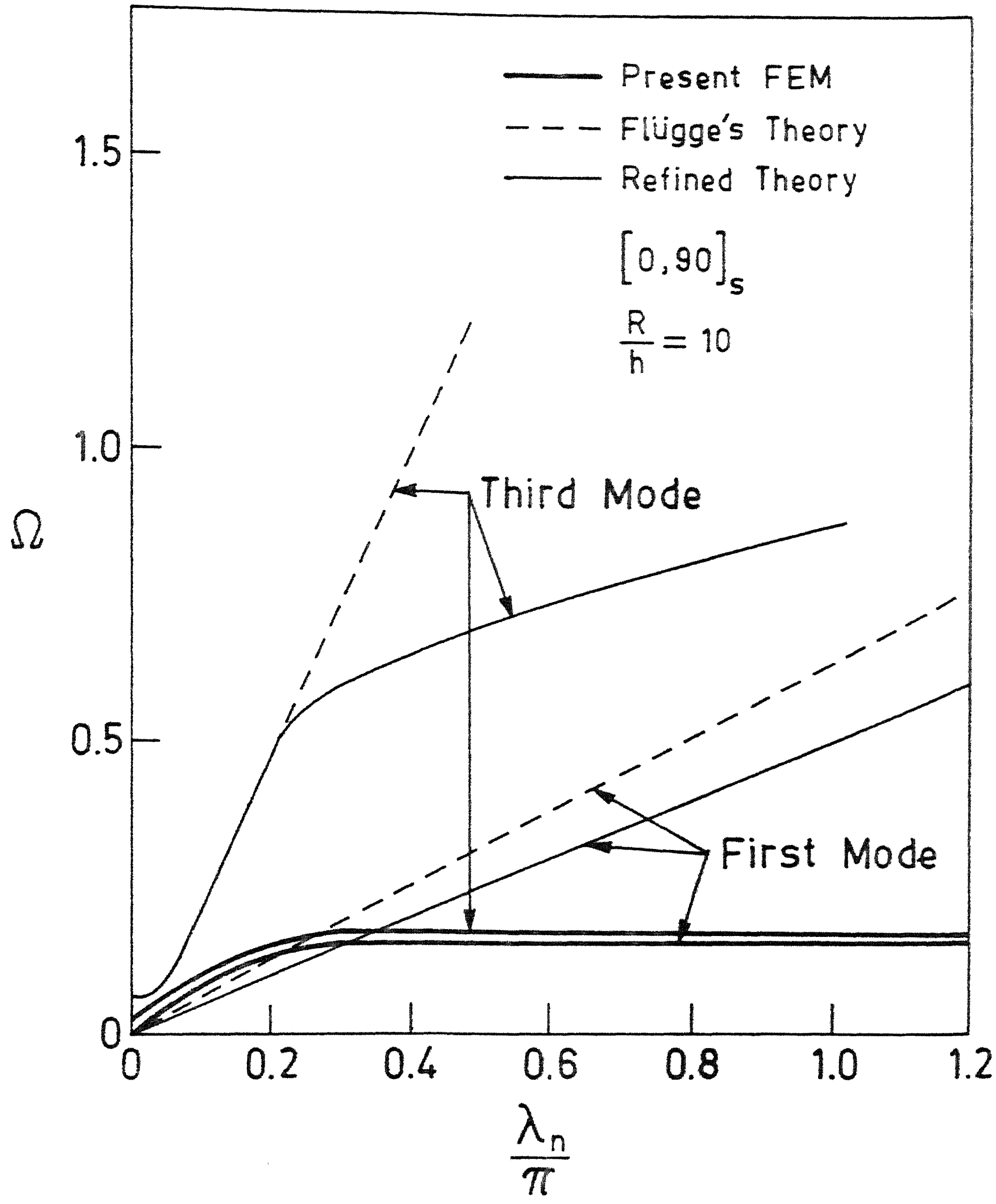


Fig. 3.8. Nondimensional Natural Frequency as a Function of h/l.

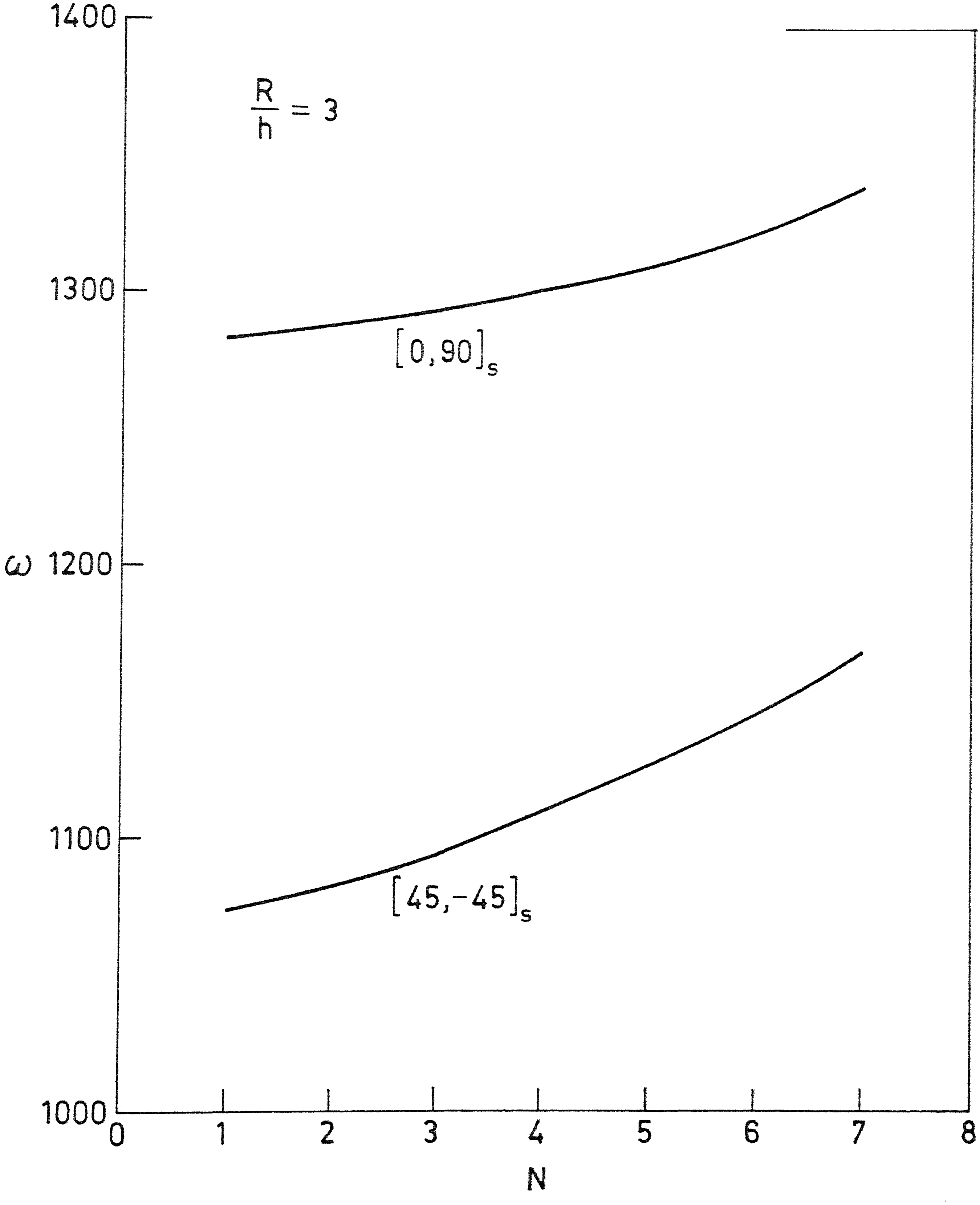


Fig. 3.9. Natural Frequency vs. Mode Number .

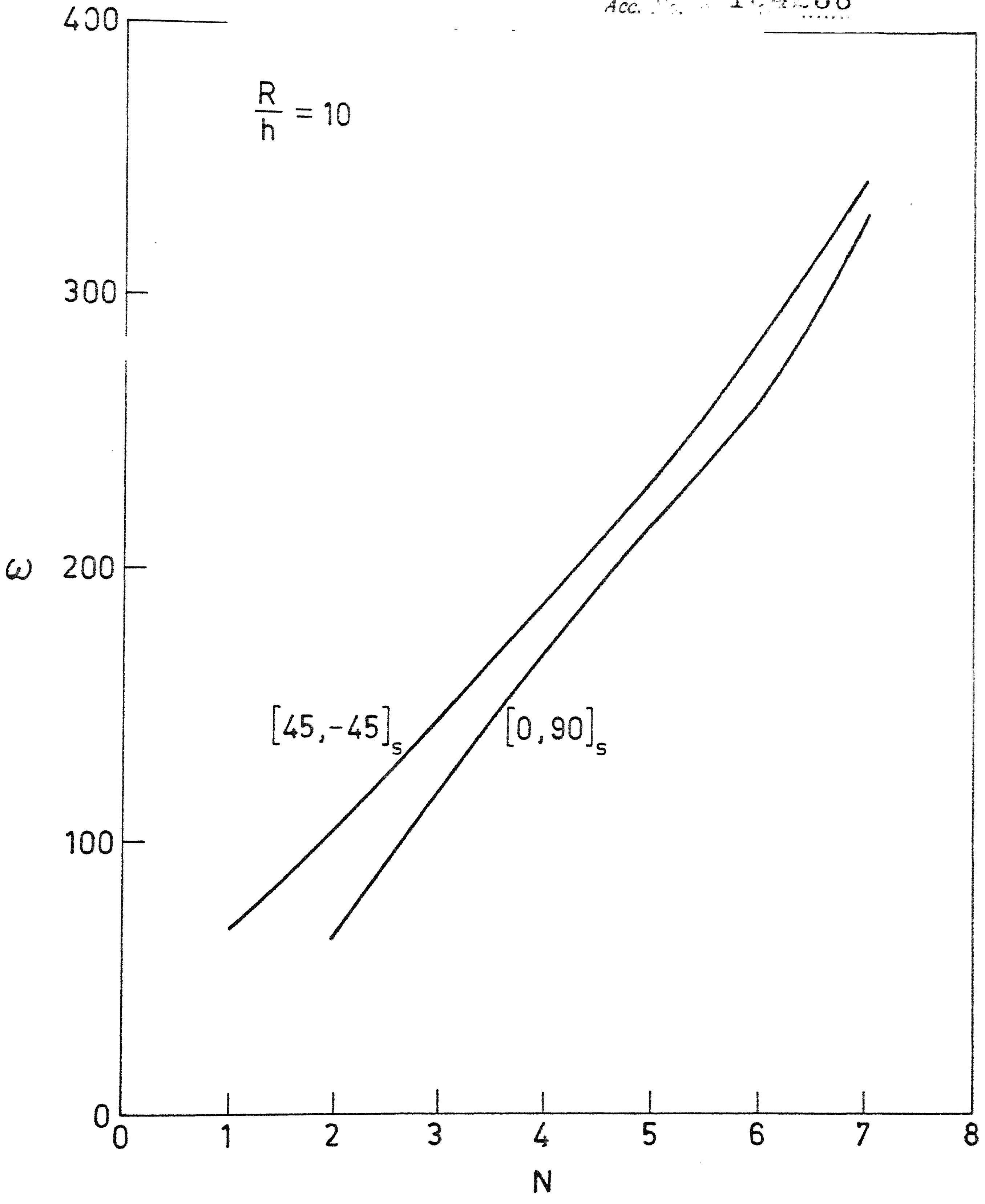


Fig. 3.10. Natural Frequency vs. Mode Number.

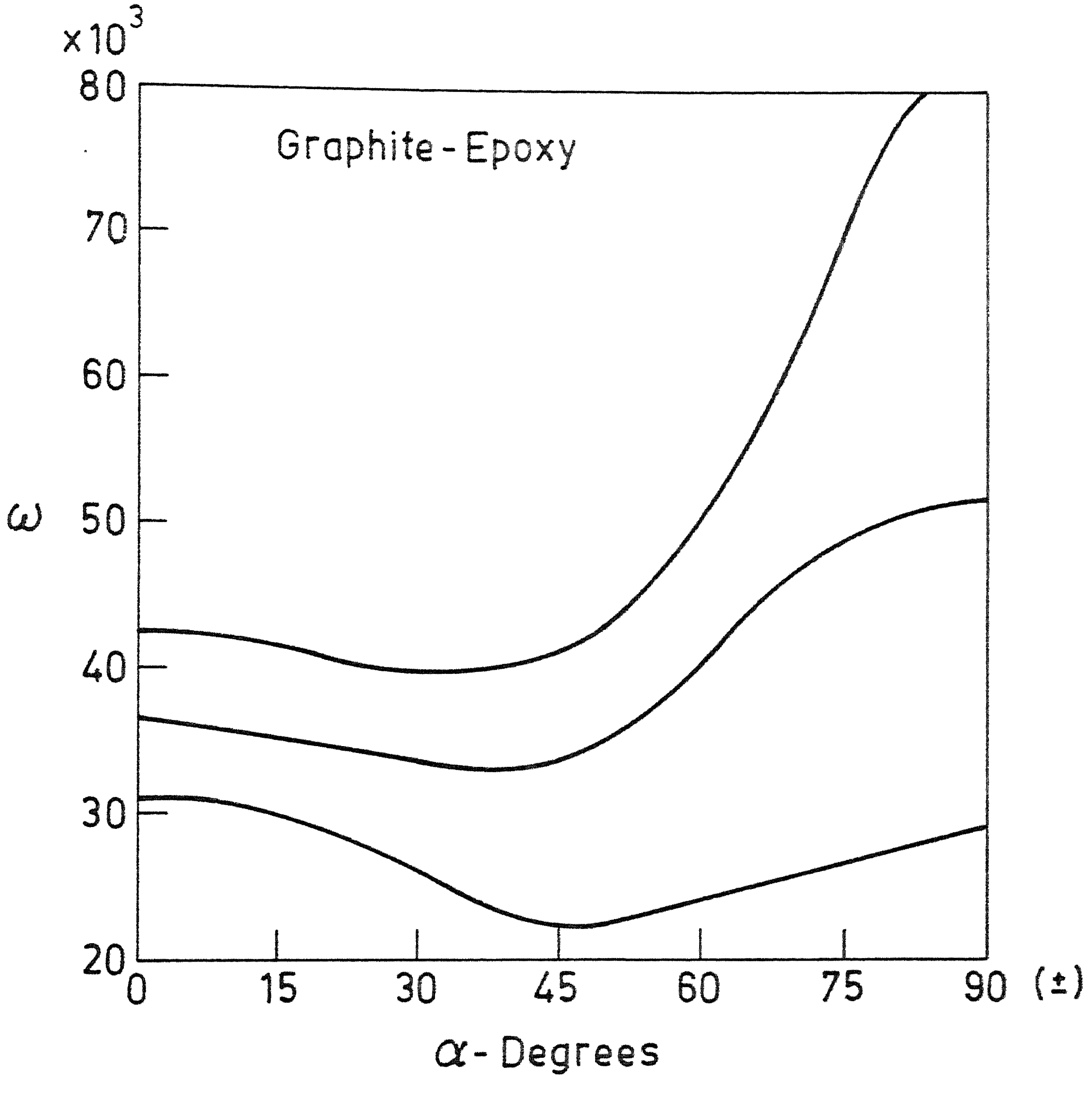


Fig. 3.11. Natural Frequency vs. Fibre Orientation.

orientation angle is further increased from [45^o] to [90^o] the frequencies are becoming larger, specially for higher modes.

The reason for this kind of behaviour is obvious when we consider the fact that the stiffness of the Fibres along longitudinal direction is higher than that in transverse direction. Therefore the composite shell with fibre orientation of [90^o] is stiffer along the circumferential direction. This does not allow the shell to expand and contract according to the axisymmetric mode shapes. And for the higher modes this resistance is substantial.

To confirm this, for [0]^o and [90]^o fibre orientation the transverse modulus of the fibres is increased from $E_L/E_T = 20$ to $E_L/E_T = 10$. The results are summarised in TABLE 3.7. From the study of this table, it can be observed that as transverse modulus E_T increases the frequency also increases. The frequencies for [90^o] fibre orientation are higher than that for [0^o] fibre orientation.

While executing the program "CURVE" for axisymmetric shells following observations can be made.

1. The lowest frequency is not the first mode frequency. This observation confirms with that of ref.[21] (page 292).
2. It is observed sometimes that for a particular mode there are two frequencies. One frequency has very low amplitude of the modal vector while for the other frequency the amplitude is higher. This is a characteristic of composite shells. Due to the bending-stretching and bending-torsion coupling these frequencies arise.

3. It is observed that bending-stretching coupling affects the mode shapes in the case of the boundary condition of simply supported shell with axial constraint.

TABLE 3.7 Effect of change in E_T. Natural Frequencies (ω) rad/sec

MODE	FIBRE ORIENTATION [0^o]		
	$\frac{E_L}{E_T} = 20$	$\frac{E_L}{E_T} = 15$	$\frac{E_L}{E_T} = 10$
1	49.62	265.11	723.07
2	92.80	305.21	740.92
3	136.05	334.82	764.01

MODE	FIBRE ORIENTATION [90^o]		
	$\frac{E_L}{E_T} = 20$	$\frac{E_L}{E_T} = 15$	$\frac{E_L}{E_T} = 10$
1	1828.65	2280.00	3420.13
2	1831.65	2282.00	3422.86
3	1836.67	2287.01	3427.39

The present FEM formulation was reduced to the assumptions of Flugge's theory by considering in-plane stresses and strains only. These results are Tabulated below.

Table 3.8 Values of Ω (No. of elements =6)

$\frac{\lambda n}{\pi}$	Flugges Theory Ω_1	Present FEM Ω_1	
0.2	0.1050	0.1089	
0.4	0.2036	0.2073	$[0,90]_S$
0.6	0.3080	0.3099	$\frac{h}{R} = \frac{1}{10}$
0.8	0.4007	0.4089	
1.0	0.5117	0.5221	
1.2	0.6220	0.6343	
0.2	0.2555	0.2918	
0.4	0.5075	0.5191	$[45,-45]_S$
0.6	0.7511	0.7633	$\frac{R}{h} = 10$
0.8	1.0193	1.0333	
1.0	1.2508	1.3233	
1.2	1.5009	1.5636	

$\frac{\lambda n}{\pi}$	Flugges Theory	Present FEM	
0.2	0.2564	0.4117	
0.4	0.5111	0.6197	$\frac{R}{h} = 3$
0.6	0.7667	0.8106	$[0,90]_S$
0.8	1.0133	1.2313	
1.0	1.3333	1.4186	
0.2	0.1511	0.2189	
0.4	0.2525	0.3331	$\frac{R}{h} = 3$
0.6	0.5010	0.6333	$[45,-46]_S$
0.8	1.0900	0.8778	
1.0	1.4500	1.1998	

CHAPTER IV

CONCLUSION

Vibration analysis is done for the laminated composite axisymmetric shells and beams.

Axisymmetric Shells

An isoparametric curved shell element with non-linear distribution of displacement and linear distribution of strain along the thickness is used. All three dimensional strains are considered.

From the analysis of the results following conclusions can be drawn.

1. Variation of frequencies with mode number n is found to be linear.
2. The frequencies for [90^o] orientation are found to be higher than that for [0^o] orientation. This behaviour is more pronounced in the case of higher modes.
3. The importance of the above-mentioned factors has been demonstrated by comparing solutions from the shell theory for contemporary engineering laminates to solultlions obtained from Flugge's theory and refined theory of ref.[2].
4. The effect of consideration of strain along thickness on the results in not substantial.

Beams

Laminate composite theory is used for the vibration

analysis of laminated composite beam. Three types of boundry conditions are considered namely;

a. Simply supported beam

b. Fixed-free beam

c. Fixed-fixed beam

The frequencies for $[0,90]_s$ laminated beam are found to be higher than that of $[0,45]_s$ laminated beam in all three types of boundry conditions.

Scope for Future Work:

In the present work, in case of beams laminated composite theory is applied. In the case of laminated axisymmetric shells, stress is given on the use of isoparametric element. The following scope for future work exists:

1. Laminated Beams
   a) Inclusion of anisotropy along the thickness.
   b) Inclusion of rotary shells
2. Laminated axisymmetric Shells
   a) Analysis of results in the case of sinusoidal forced vibrations.
   b) Analysis of results in the case of conical shells, shells with Gaussian curvatures and other axisymmetric shells, spherical caps etc.

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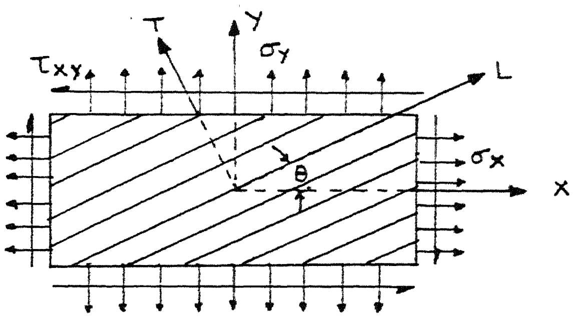
APPENDIX - I

An orthotropic lamina with principal axes oriented at angle θ with the reference axes is shown in figure.

$$\begin{bmatrix} \sigma_L \\ \sigma_T \\ \tau_{LT} \end{bmatrix} = [T] \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}$$

$$\begin{bmatrix} \varepsilon_L \\ \varepsilon_T \\ \frac{1}{2}\gamma_{LT} \end{bmatrix} = [T] \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \frac{1}{2}\gamma_{xy} \end{bmatrix}$$



where [T] is transformation matrix given by

$$[T] = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta \times \cos\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta \times \cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}$$

Also,

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = [T]^{-1} \begin{bmatrix} \sigma_L \\ \sigma_T \\ \tau_{LT} \end{bmatrix}$$

As,

$$\begin{bmatrix} \sigma_L \\ \sigma_T \\ \tau_{LT} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & 2Q_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_L \\ \varepsilon_T \\ \frac{1}{2}\gamma_{LT} \end{bmatrix}$$

therefore,

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = [T]^{-1} [\,Q\,] [T] \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \frac{1}{2}\gamma_{xy} \end{bmatrix}$$

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}$$

where,

$$\bar{Q}_{11} = Q_{11} \cos^4\theta + Q_{22} \sin^4\theta + 2(Q_{12}+2Q_{66}) \sin^2\theta \cos^2\theta$$

$$\bar{Q}_{22} = Q_{11} \sin^4\theta + Q_{22} \cos^4\theta + 2(Q_{12}+2Q_{66}) \sin^2\theta \cos^2\theta$$

$$\bar{Q}_{12} = (Q_{11}+Q_{22}-4Q_{66}) \sin^2\theta \cos^2\theta + Q_{12} (\cos^4\theta + \sin^4\theta)$$

$$\bar{Q}_{66} = (Q_{11}+Q_{22} - 2Q_{12} - 2Q_{66}) \sin^2\theta \cos^2\theta + Q_{66}(\sin^4\theta+\cos^4\theta)$$

$$\bar{Q}_{16} = (Q_{11}-Q_{12}-2Q_{66})\cos^3\theta\sin\theta - (Q_{22}-Q_{12}-2Q_{66}) \cos\theta \sin^3\theta$$

$$\bar{Q}_{26} = (Q_{11}-Q_{12}-2Q_{66}) \cos\theta \sin^3\theta - (Q_{22}-Q_{12}-2Q_{66})\cos^3\theta \sin\theta$$

APPENDIX -II

The present software PROJECT: FORT is based on the ND-560 system. The FORTRAN-77 Language is used for the programming. The software uses available NAG and GKS library subroutines.

The various modules connected to the main program PROJECT: FORT are as shown in Fig. 3.1. These are :

(1) INCUR :

This subroutine while executing, collects the data from the user which is used as input data for the another module CURVE. The following data is required :

1. The number of elements for the axisymmetric shell
2. The geometry of the element described by the nodal points in the Y-R plane as shown in Fig. 2.5.
3. Thickness of the element at each node
4. Material properties as follows :

E_L Longitudinal modulus of the composite

E_T Transverse modulus of the composite

G_{LT} Shear modulus in L-T plane

G_{TT} Shear modulus in T-T plane

ν_{LT} Poisson's ratio in L-T direction

ν_{TT} Poisson's ratio in T-T direction

5. Number of layers in the composite in the thickness.
6. Angle of orientation for each layer and the thickness.
7. Density of composite.
8. Boundary conditions for shell.
9. Forcing frequency in case of forced sinusoidal vibration.
10. The nodal force vector.

The file CURVE : INP is written by the execution of INCUR.

(II) CURVE :

This subroutine is for the vibration analysis of laminated Fibre Reinforced Plastic composite axi-symmetric shells. The working of subroutine is as follows :

(1) [s] matrix is calculated according to equation (2.25)

(2) $[s]^{-1}$ matrix is calculated using (F01AAF) NAG subroutine.

(3) Then according to equation (2.29) and (2.30) the elemental stiffness and mass matrices are calculated. The subroutine SHAPE or SHAP is called according to the condition of

1. The strain along thickness is neglected - Module CURVE
2. The strain along thickness is considered - Module ZETACUR.

The subroutines SHAPE or SHAPE calculates the matrix $[B]^T [D] [B]$ using the shape functions.

(4) The subroutine BOUND assigns the boundary conditions for the Global mass and stiffness matrices. It also decides their size by following expression.

NX= (2x NOE +1) x 4 - KBC

where

NX = total number of degrees of freedom

NOE = number of elements

KBC = known boundary conditions

(5) The subroutine ASSEMB assembles the elemental stiffness and mass matrices into the Global stiffness and mass matrices according to the boundary conditions.

(6) The subroutine (F02AEF) is called from NAG library. This subroutine solves equation (2.31). The output is stored in files CURVE : OUT . Eigen values and vectors are obtained as output.

(7) After selection of particular frequency by user the vectors are saperated for this frequency. This is done by using boundary conditions.

(8) These vectors give the displacement at each node. These give mode shapes which are then drawn using GKS subroutines.

(9) Subroutine FORCE is for the sinusoidal forced vibration response. This subroutine solves the equation (2.12) using the subroutine (F01AAF) from NAG library.

The response obtained is then processed and drawn on the screen using GKS.

The module CURVE was executed for the input data given below :

$E_L = 20\times 10^6$ psi $\quad \nu_{LT} = \nu_{TT} = 0.25$

$E_T = 1\times 10^6$ psi $\quad \frac{L}{R} = 0.5$, $\frac{R}{h} = 10$

$G_{LT} = 0.6\times 10^6$ psi

$G_{TT} = 0.5\times 10^6$ psi $\quad [0,90]_s$

The mode shapes obtained are shown in Fig. 1. The normalization factor is 2.73657×10^{-3}.

For the sinusoidal forced vibration problem the following input is given:

Concentrated force at node 1 of each element except the first element $= 2\times 10^{11}$ lb.

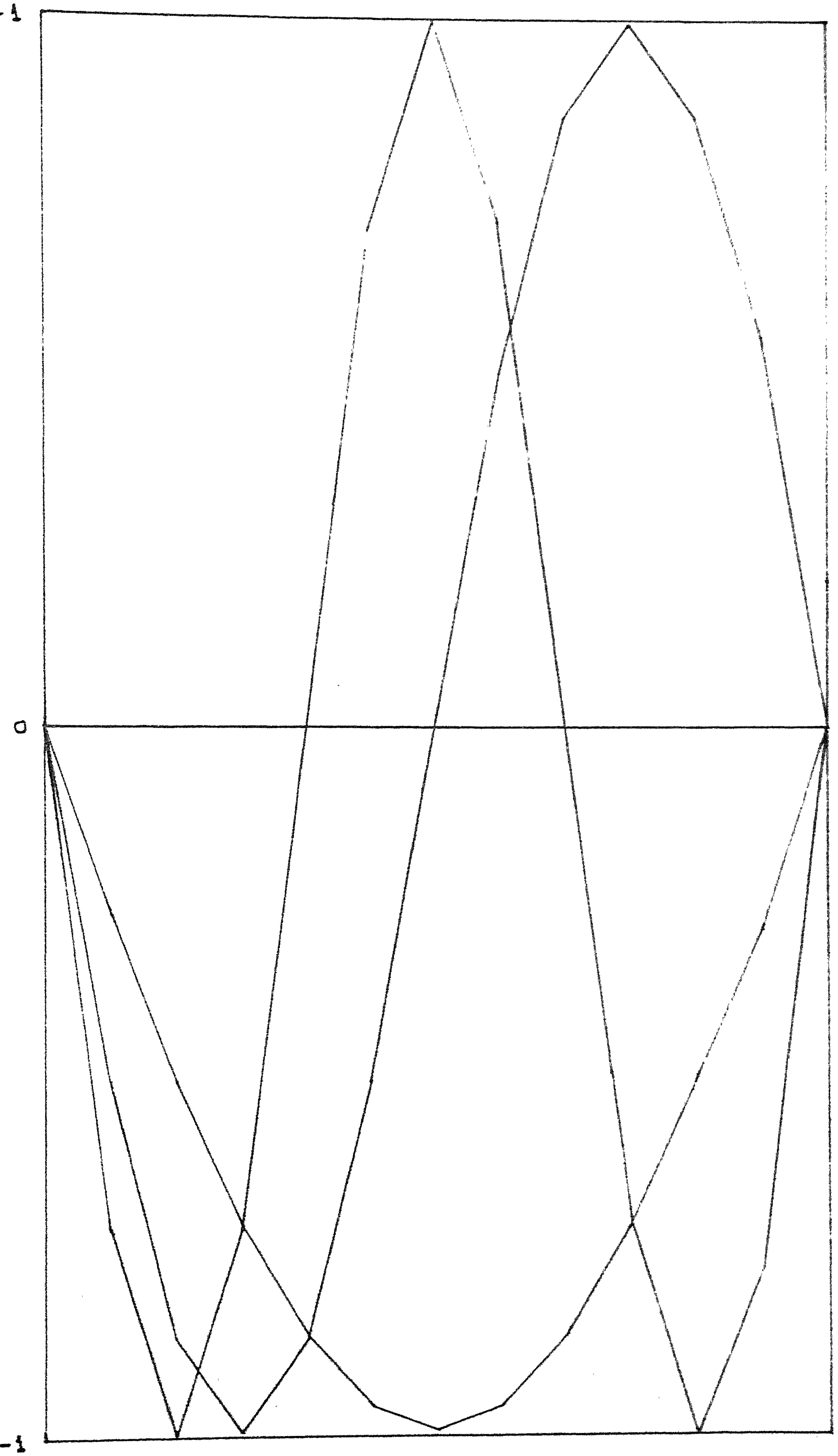


Fig. 1. MODE SHAPES

Frequency of forced vibration = $2x\ 10^4$ rad/sec.

The force is in positive R-direction.
The response in θ and Y direction is negligible (10^{-17}-10^{-11}).
The response in the R-direction is as shown in Fig. 2.

(III) ZETACUR :

This module works like CURVE except that it considers the strain along the thickness direction of the shell.

(IV) FV BEAM :

The execution of module FV BEAM leads to calling the subroutine FV BEAM. This subroutine is for the vibration analysis of laminated composite beams. The working of this subroutine is as follows :

(1) The input data is collected from the user. This data consists of the following items :
   (i) The length, breadth, height of beam.
   (ii) The material properties of the composite laminate.
   (iii) Number of finite elements.
   (iv) Boundary conditions.

(2) Subroutine BOUNDER is called for assigning the boundary conditions. The size of the global matrix is decided by the following expression :

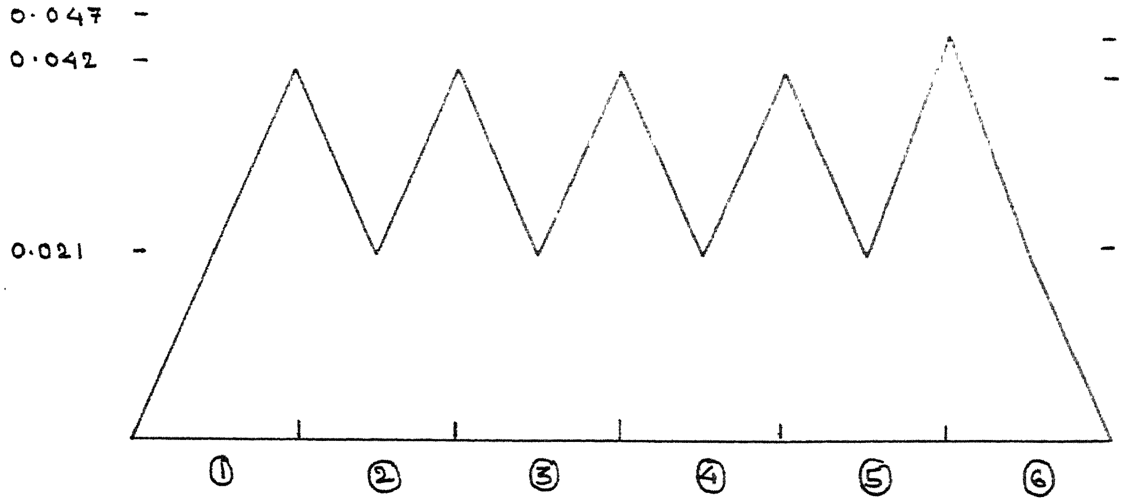


Fig. 2. RESPONSE IN R- DIRECTION

NX = (FELENO +1)x2 - known boundary conditions.

where, NX = total degrees of freedom

FELENO = number of finite elements.

(3) Subroutine ASSMBL is called for assembling the elemental mass and stiffness matrices into global mass and stiffness matrices.

(4) Subroutine FO2AEF is called to solve the eigen value problem from NAG library.

(5) Subroutine FORSIN is called for the sinusoidal forced vibration problem which uses FOIAAF subroutine from NAG library to solve equation (2.12).

(V) ST BEAM :

This module calls subroutine ST BEAM . This subroutine takes the input data as :

(i) Geometrical dimensions of beam

(ii) Material properties of the composite laminate

(iii) Loading types

(iv) Boundary conditions

(v) Point at which the stresses and strains are to be found out.

(vi) Points at which the bending moment and shear forces are to be calculated.

The various subroutines are for the different loading conditions and boundary conditions. Exact analysis is done.

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